

# A Practical Method to Solve Cut-off Coulomb Problems in the Momentum Space

— Application to the Lippmann-Schwinger Resonating-Group Method and the  $pd$  Elastic Scattering —

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A practical method to solve cut-off Coulomb problems of two-cluster systems in the momentum space is given. When a sharply cut-off Coulomb force with a cut-off radius  $\rho$  is introduced at the level of constituent particles, two-cluster direct potential of the Coulomb force becomes in general a local screened Coulomb potential. The asymptotic Hamiltonian yields two types of asymptotic waves; one is an approximate Coulomb wave with  $\rho$  in the middle-range region, and the other a free (no-Coulomb) wave in the longest-range region. The constant Wronskians of this Hamiltonian can be calculated in either region. We can evaluate the Coulomb-modified nuclear phase shifts for the screened Coulomb problem, using the matching condition proposed by Vincent and Phatak for the sharply cut-off Coulomb problem. We apply this method first to an exactly solvable model of the  $\alpha\alpha$  scattering with the Ali-Bodmer potential and confirm that a complete solution is obtained with a finite  $\rho$ . The stability of nuclear phase shifts with respect to the change of  $\rho$  in some appropriate range is demonstrated in the  $\alpha\alpha$  resonating-group method (RGM) using the Minnesota three-range force. An application to the  $pd$  elastic scattering is also discussed.

Subject Index: 200, 205

## §1. Introduction

In the momentum representation, incorporation of the long-range Coulomb force always poses problems. In particular, three-body scattering problems involving the Coulomb force are still under intensive investigations.<sup>1),2),3),4),5)</sup> Here we mainly consider a much simpler problem of solving the Lippmann-Schwinger (LS) equations in the momentum representation, in which the Coulomb force is included in the two-cluster resonating-group method (RGM). In this particular case, the longest range direct potential consists of a nuclear direct potential and the long-range Coulomb potential in the error function form when simple harmonic-oscillator shell-model wave functions are employed for clusters. We introduce a sharp cut-off radius  $\rho$  for the Coulomb force acting between constituent particles. We can solve the LS equations and obtain the  $T$ -matrix in the standard procedure. A problem is how to extract the correct nuclear phase shifts from this  $T$ -matrix or the phase shifts, including the effect of the screened Coulomb force. Here we propose a simple method, taking examples of the  $\alpha\alpha$  RGM and the proton-deuteron ( $pd$ ) elastic scattering using the quark-model baryon baryon interaction.

The standard procedure to solve the Coulomb problem, including the short-range nuclear potential and the long-range Coulomb force is well established as long as two-body problems are concerned. In the treatment in terms of the distorted waves, the relative-wave function  $\psi_\ell(r)$  between two clusters are solved numerically in the

configuration space, including the complete Coulomb force. The nuclear phase shift  $\delta_\ell^N$  is then obtained from the asymptotic form of the relative wave function through the so-called matching condition

$$\tan \delta_\ell^N = -\frac{W[F_\ell, \psi_\ell]_\rho}{W[G_\ell, \psi_\ell]_\rho}, \quad (1.1)$$

where  $F_\ell = F_\ell(k, r)$  and  $G_\ell = G_\ell(k, r)$  stand for the regular and irregular Coulomb wave functions, respectively, and  $W[f, g] = f(k, r)(\partial/\partial r)g(k, r) - g(k, r)(\partial/\partial r)f(k, r)$  is the Wronskian for functions  $f(k, r)$  and  $g(k, r)$ . The Wronskian values in Eq. (1.1) are evaluated at the relative distance  $r = \rho$ , which should be taken large enough to avoid the effect of the nuclear force in the short-range region. Quite naturally, this standard procedure should be modified in the momentum representation, if we try to solve three-body problems like the  $pd$  scattering, and also the Lippmann-Schwinger RGM (LS-RGM) equations with the Coulomb interaction. In these applications, the basic ingredient is the  $T$ -matrix, which is usually formulated in the momentum space. The Born term of the  $T$ -matrix is already singular for the diagonal part of the initial and final momenta,  $q_f = q_i$ .

A practical method to deal with the Coulomb force in the momentum representation is to use the cut-off or screened Coulomb force. In the early work by Vincent and Phatak,<sup>6)</sup> the Coulomb force in the  $\pi^\pm + {}^{16}\text{O}$  scattering is assumed to be a sharply cut-off Coulomb force

$$\omega_\rho(r) = \frac{2\eta k}{r} \theta(\rho - r), \quad (1.2)$$

where  $\eta = \alpha/\hbar v$  is the Sommerfeld parameter and  $\theta$  is the step function. Since the relative wave function has a Coulomb-free asymptotic behavior, the asymptotic wave is composed of the nuclear plus cut-off Coulomb phase shift and the known Bessel and Neumann functions. This phase shift  $\bar{\delta}_\ell^\rho$  is obtained by solving the LS equation for the  $T$ -matrix in the momentum space. The nuclear phase shift  $\delta_\ell^N$  is then calculated from the matching condition of the asymptotic waves:

$$\tan \delta_\ell^N = -\frac{W[F_\ell, u_\ell]_\rho + \tan \bar{\delta}_\ell^\rho W[F_\ell, v_\ell]_\rho}{W[G_\ell, u_\ell]_\rho + \tan \bar{\delta}_\ell^\rho W[G_\ell, v_\ell]_\rho}, \quad (1.3)$$

with the sufficiently large  $\rho$ .

A recent Coulomb treatment by Deltuva et al.<sup>7),8),9)</sup> for the  $pd$  scattering uses a screened Coulomb potential in the form of

$$\omega_\rho(r) = \frac{2\eta k}{r} e^{-(r/\rho)^n} \quad (1.4)$$

with  $n \sim 4$ , and the “screening and renormalization procedure”, which is developed by Alt et al.<sup>10),11),12),13),14)</sup> The basic ingredient of this approach is the Taylor’s theorem,<sup>15),16)</sup> which implies that the phase shift of the screened Coulomb potential  $\delta_\ell^\rho$  requires the renormalization  $\delta_\ell^\rho \longrightarrow \sigma_\ell - \zeta^\rho(k)$  as  $\rho \rightarrow \infty$  in the sense of

distribution, where

$$\zeta^\rho(k) = \frac{1}{2k} \int_{\frac{1}{2k}}^{\infty} \omega_\rho(r) dr \quad (1.5)$$

is the diverging renormalization phase determined from the explicit form of the screened Coulomb potential  $\omega_\rho(r)$ . Since the relative wave functions with the screened Coulomb potential always suffer the renormalization of this phase factor, the complete  $pd$  scattering amplitude is achieved only when the limit  $\rho \rightarrow \infty$  is reached in the two-potential formula for the scattering amplitude. In practice, this limit is taken numerically such that the well converged result is obtained. A problem of this procedure is that the error estimate of the finite  $\rho$  is not possible, and we usually need to take a very large  $\rho$  value, for which solving the Alt-Grassberger-Sandhas equation (AGS equation)<sup>17)</sup> accurately is not easy because of the quasi-singular nature of the screened Coulomb potential. The convergence of the partial wave decomposition also becomes problematic, if the Coulomb singularity is so strong.

A final goal of this study is to find an approximate but practical method to incorporate the Coulomb force to the  $pd$  elastic scattering, by using a reasonable magnitude of  $\rho$ . For this purpose, we incorporate the Vincent and Phatak approach<sup>6)</sup> to the “screening and renormalization procedure”. In this paper, we first consider a simple potential model for the  $\alpha\alpha$  scattering and examine if this approach gives a reasonable accuracy of the phase shift, using the screened Coulomb potential. The stability of the nuclear phase shift with respect to the change of  $\rho$  in an appropriate range is examined by  $\alpha\alpha$  LS-RGM. An application to the  $pd$  elastic scattering is briefly discussed.

In the next section, we discuss the sharply cut-off Coulomb problem, for which analytic derivation of the cut-off Coulomb wave functions is feasible. The definitions of the pure Coulomb wave functions used in this paper are gathered in Appendix A. A general procedure to calculate the nuclear phase shift from solutions of the LS equations for the two-cluster  $T$ -matrix is discussed in §3. In §4, a formulation for the screened Coulomb problem is given, by paying an attention to new features appearing in the screened Coulomb potential. An extension to deal with the  $pd$  elastic scattering in the present approach is given in §5. In §6, numerical performance is examined, first for an exactly solvable model in the case of Ali-Bodmer’s phenomenological  $\alpha\alpha$  potential, secondly for the  $\alpha\alpha$  LS-RGM using the Minnesota three-range force, and finally for the  $pd$  elastic scattering using the quark-model baryon-baryon interaction fss2. In Appendix B, shift functions of various screening functions are evaluated. The screening function  $\alpha^\rho(R)$  for the  $pd$  scattering is derived in Appendix C. The last section is devoted to a summary and outlook.

## §2. Exact solutions of the sharply cut-off Coulomb problem

In this section, we assume a sharply cut-off Coulomb potential Eq. (1.2) and consider the pure Coulomb problem as the limit of  $\rho \rightarrow \infty$ . The regular solution  $\varphi_\ell^\rho(k, r)$

corresponding to the Jost solution satisfies the following integral equation:<sup>18),19),20)</sup>

$$\varphi_\ell^\rho(k, r) = \frac{1}{k^{\ell+1}} u_\ell(kr) + \int_0^r \mathcal{G}_{0\ell}(r, r'; k) \frac{2k\eta}{r'} \theta(\rho - r') \varphi_\ell^\rho(k, r') dr' . \quad (2.1)$$

Here,  $u_\ell(kr)$  is the Riccati Bessel function and the Green function  $\mathcal{G}_{0\ell}(r, r'; k)$  is given by

$$\mathcal{G}_{0\ell}(r, r'; k) = \frac{1}{k} [u_\ell(kr) v_\ell(kr') - v_\ell(kr) u_\ell(kr')] \theta(r - r') , \quad (2.2)$$

with  $v_\ell(kr)$  being the Riccati Neumann function. For  $r \leq \rho$ ,  $\varphi_\ell^\rho(k, r)$  is the same as the regular Coulomb function  $\varphi_\ell(k, r)$  given in Eq. (A.1), which implies

$$\begin{aligned} \frac{1}{k^\ell} F_\ell^\rho(k) \psi_\ell^\rho(k, r) &= \frac{1}{k^\ell} F_\ell(k) \psi_\ell(k, r) \quad \rightarrow \quad \psi_\ell^\rho(k, r) = \frac{F_\ell(k)}{F_\ell^\rho(k)} \psi_\ell(k, r) \\ \text{for } r &\leq \rho , \end{aligned} \quad (2.3)$$

where  $F_\ell^\rho(k)$  is the Jost function of the sharply cut-off Coulomb potential and  $F_\ell(k)$  the Coulomb Jost function. If we use this in the integral equation for the regular solution  $\psi_\ell^\rho(k, r)$ ,

$$\psi_\ell^\rho(k, r) = \frac{1}{k} u_\ell(kr) + \langle r | G_{0\ell} \omega_\rho \psi_\ell^\rho \rangle , \quad (2.4)$$

we obtain

$$\psi_\ell(k, r) = \frac{F_\ell^\rho(k)}{F_\ell(k)} \frac{1}{k} u_\ell(kr) + \langle r | G_{0\ell} \omega_\rho \psi_\ell \rangle \quad \text{for } r \leq \rho , \quad (2.5)$$

with  $G_{0\ell}$  being the regular Green function. The Jost function  $F_\ell^\rho(k)$  of the sharply cut-off Coulomb potential is calculated from  $F_\ell^\rho(k) = 1 + k^\ell \langle \omega_\ell^{(-)} | \omega_\rho | \varphi_\ell^\rho \rangle$  as<sup>21)</sup>

$$\begin{aligned} F_\ell^\rho(k) &= -i\eta \frac{\ell!}{\Gamma(\ell+1-i\eta)} \sum_{n=0}^{\ell} \frac{(\ell+n)!}{n!} \sum_{m=0}^{\ell-n} \frac{(-2ik\rho)^m}{m!} \frac{\Gamma(n+m-i\eta)}{\Gamma(\ell+n+m+1)} \\ &\quad \times F(n+m-i\eta, \ell+n+m+1, 2ik\rho) . \end{aligned} \quad (2.6)$$

In particular,  $S$ -wave Jost function is very simple:

$$F_0^\rho(k) = F(-i\eta, 1, 2ik\rho) . \quad (2.7)$$

If we use the asymptotic form of the confluent hypergeometrical function  $F(\alpha, \gamma, z)$  at  $|z| \rightarrow \infty$ , we obtain

$$\lim_{\rho \rightarrow \infty} (2k\rho)^{-i\eta} F_\ell^\rho(k) = F_\ell(k) = e^{\pi\eta/2} \frac{\ell!}{\Gamma(\ell+1+i\eta)} , \quad (2.8)$$

resulting in

$$\lim_{\rho \rightarrow \infty} \frac{F_\ell^\rho(k)}{F_\ell(k)} (2k\rho)^{-i\eta} = 1 . \quad (2.9)$$

This relationship yields the limit of Eq. (2.5) as

$$\psi_\ell(k, r) = \lim_{\rho \rightarrow \infty} \left\{ \frac{1}{k} u_\ell(kr) (2k\rho)^{i\eta} + \langle r | G_{0\ell} \omega_\rho \psi_\ell \rangle \right\} . \quad (2.10)$$

Furthermore, Eq. (2.3) implies

$$\lim_{\rho \rightarrow \infty} (2k\rho)^{i\eta} \psi_\ell^\rho(k, r) = \psi_\ell(k, r) \quad (2.11)$$

The asymptotic behavior of the cut-off Coulomb phase shift for  $\rho \rightarrow \infty$  can be derived from the non-Coulomb version of Eq. (1.1), since Wronskians  $W[u_\ell, \psi_\ell]_\rho$  and  $W[v_\ell, \psi_\ell]_\rho$  with the  $\rho \rightarrow \infty$  limit are analytically calculated. The result is, of course,

$$\delta_\ell^\rho \rightarrow \sigma_\ell - \eta \log 2k\rho \quad \text{as} \quad \rho \rightarrow \infty , \quad (2.12)$$

with the ambiguity of integral multiples of  $\pi$ .<sup>15), 16)</sup>

The Jost solution for the sharply cut-off Coulomb potential is defined by the integral equation

$$f_\ell^\rho(k, r) = \omega_\ell^{(+)}(kr) + \int_r^\infty g_{0\ell}(r, r'; k) \frac{2k\eta}{r'} \theta(\rho - r') f_\ell^\rho(k, r') dr' , \quad (2.13)$$

where the Green function is

$$g_{0\ell}(r, r'; k) = -\frac{1}{k} [u_\ell(kr) v_\ell(kr') - v_\ell(kr) u_\ell(kr')] \theta(r' - r) . \quad (2.14)$$

The asymptotic behavior is given by

$$f_\ell^\rho(k, r) = \omega_\ell^{(+)}(kr) \sim e^{i(kr - (\pi/2)\ell)} \quad \text{for} \quad r \geq \rho . \quad (2.15)$$

For the Coulomb solutions, we cannot formulate the integral equation, since the asymptotic behavior is different from Eq. (2.15). However, for  $r \leq \rho$ ,  $f_\ell^\rho(k, r)$  can be written as a linear combination of two independent Coulomb Jost solutions,  $f_\ell(k, r)$  and  $f_\ell^*(k, r)$ :

$$f_\ell^\rho(k, r) = C_1^\rho f_\ell(k, r) + C_2^\rho f_\ell^*(k, r) \quad \text{for} \quad r \leq \rho . \quad (2.16)$$

The coefficients,  $C_1^\rho$  and  $C_2^\rho$  are derived by evaluating the Wronskians  $W[f_\ell^*, f_\ell^\rho]$  and  $W[f_\ell, f_\ell^\rho]$  at  $\rho \rightarrow \infty$ . We find

$$\begin{aligned} C_1^\rho &= \left( 1 - \frac{\eta}{2k\rho} \right) (2k\rho)^{i\eta} , \\ C_2^\rho &= (-)^{\ell+1} \frac{\eta}{2k\rho} (2k\rho)^{-i\eta} e^{2ik\rho} \quad \text{as} \quad \rho \rightarrow \infty . \end{aligned} \quad (2.17)$$

Thus, if we use the symmetry Eq. (A.5) for  $f_\ell^*(k, r)$ , we find

$$\begin{aligned} f_\ell^\rho(k, r) &= \left( 1 - \frac{\eta}{2k\rho} \right) (2k\rho)^{i\eta} f_\ell(k, r) - \frac{\eta}{2k\rho} e^{\pi\eta} (2k\rho)^{-i\eta} e^{2ik\rho} f_\ell(-k, r) \\ &\sim (2k\rho)^{i\eta} f_\ell(k, r) \quad \text{for} \quad r \leq \rho \rightarrow \infty . \end{aligned} \quad (2.18)$$

After all, we have obtained

$$\lim_{\rho \rightarrow \infty} (2k\rho)^{-i\eta} f_\ell^\rho(k, r) = f_\ell(k, r) . \quad (2.19)$$

Note that the renormalization phase is the complex conjugate of the one appearing in Eq. (2.11). This results in a basic property of the sharply cut-off Coulomb potential that the Coulomb Green function can be obtained as the  $\rho \rightarrow \infty$  limit of the sharply cut-off Coulomb Green function. Namely, if we define

$$\begin{aligned} G_\ell^\rho(r, r'; k) &= -\psi_\ell^\rho(k, r_<) f_\ell^\rho(k, r_>) \\ G_\ell^C(r, r'; k) &= -\psi_\ell(k, r_<) f_\ell(k, r_>) , \end{aligned} \quad (2.20)$$

then we find

$$\lim_{\rho \rightarrow \infty} G_\ell^\rho(r, r'; k) = G_\ell^C(r, r'; k) \quad \text{for } r, r' \leq \rho \rightarrow \infty . \quad (2.21)$$

This relationship is valid only when the Green functions are operated on the short-range potentials.

One can derive the Coulomb scattering amplitude from the scattering amplitude for the sharply cut-off Coulomb potential. We use the formula for the short range force

$$\begin{aligned} f_\ell^\rho &= -\frac{1}{k} \langle u_\ell | \omega_\rho | \psi_\ell^\rho \rangle = -\frac{1}{k^2} \langle u_\ell | T_\ell^\rho | u_\ell \rangle = -\frac{1}{k} \frac{\Im m F_\ell^\rho(k)}{F_\ell^\rho(k)} \\ &= \frac{1}{2ik} \left( \frac{F_\ell^\rho(k)^*}{F_\ell^\rho(k)} - 1 \right) , \end{aligned} \quad (2.22)$$

and calculate

$$f_\ell \equiv \lim_{\rho \rightarrow \infty} (2k\rho)^{i\eta} f_\ell^\rho (2k\rho)^{i\eta} . \quad (2.23)$$

Equation (2.8) yields for  $\rho \rightarrow \infty$

$$\begin{aligned} f_\ell &\sim \frac{1}{2ik} \left( \frac{(F_\ell^\rho(k)(2k\rho)^{-i\eta})^*}{F_\ell^\rho(k)(2k\rho)^{-i\eta}} - (2k\rho)^{2i\eta} \right) \\ &= \frac{1}{2ik} \left( \frac{F_\ell(k)^*}{F_\ell(k)} - (2k\rho)^{2i\eta} \right) = \frac{1}{2ik} (e^{2i\sigma_\ell} - (2k\rho)^{2i\eta}) \\ &= \frac{1}{2ik} (e^{2i\sigma_\ell} - 1) - \frac{1}{2ik} ((2k\rho)^{2i\eta} - 1) = f_\ell^C - \frac{1}{2ik} ((2k\rho)^{2i\eta} - 1) . \end{aligned} \quad (2.24)$$

Here, the last term is  $\ell$ -independent and contributes only to  $\theta = 0$  if we add up over all the partial waves. Thus, we find

$$\begin{aligned} f(\theta) &= \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell P_\ell(\cos \theta) \\ &= f^C(\theta) - \sum_{\ell=0}^{\infty} (2\ell+1) \frac{1}{2ik} ((2k\rho)^{2i\eta} - 1) P_\ell(\cos \theta) \\ &= f^C(\theta) \quad \text{for } \theta \neq 0 . \end{aligned} \quad (2.25)$$

Here,

$$\begin{aligned} f^C(\theta) &= \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1) (e^{2i\sigma_\ell} - 1) P_\ell(\cos \theta) \\ &= -\frac{\eta}{2k \left(\sin \frac{\theta}{2}\right)^2} e^{-2i\eta \log \left(\sin \frac{\theta}{2}\right)} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \end{aligned} \quad (2.26)$$

is the standard Coulomb scattering amplitude.

We should note that the renormalization phase  $(2k\rho)^{i\eta}$  appearing in the above equations is nothing but the Taylor's phase factor Eq. (1.5). In fact, we can easily show for Eq. (1.2)

$$\zeta^\rho(k) = \frac{1}{2k} \int_{\frac{1}{2k}}^{\infty} \omega_\rho(r) dr = \eta \int_{\frac{1}{2k}}^{\rho} \frac{1}{r} dr = \eta \log(2k\rho) . \quad (2.27)$$

Then, the relationship in Eqs. (2.11) and (2.19) can be written as

$$\lim_{\rho \rightarrow \infty} e^{i\zeta^\rho(k)} \psi_\ell^\rho(k, r) = \psi_\ell(k, r) , \quad \lim_{\rho \rightarrow \infty} e^{-i\zeta^\rho(k)} f_\ell^\rho(k, r) = f_\ell(k, r) . \quad (2.28)$$

More basically, different parametrizations of Coulomb functions in Eq. (A.9) and a trivial relationship

$$\psi_\ell^{(+)}(k, r) = \frac{1}{k} \Im f_\ell(k, r) + f_\ell^C f_\ell(k, r) \quad (2.29)$$

derived from them are essential. Since many relations are also valid even for more general screened Coulomb functions introduced in §4, we reformulate the sharply cut-off Coulomb problem in more general form, using the parametrization of wave functions as

$$\begin{aligned} \psi_\ell^\rho(k, r) &= \frac{1}{k} e^{i\delta_\ell^\rho} F_\ell^\rho(k, r) , \\ \varphi_\ell^\rho(k, r) &= \frac{1}{k^{\ell+1}} |F_\ell^\rho(k)| F_\ell^\rho(k, r) = \text{real} , \\ f_\ell^\rho(k, r) &= e^{-i\delta_\ell^\rho} [G_\ell^\rho(k, r) + iF_\ell^\rho(k, r)] , \\ f_\ell^{\rho*}(k, r) &= e^{i\delta_\ell^\rho} [G_\ell^\rho(k, r) - iF_\ell^\rho(k, r)] . \end{aligned} \quad (2.30)$$

For the sharply cut-off Coulomb force, the basic screened Coulomb wave functions satisfying

$$\lim_{\rho \rightarrow \infty} F_\ell^\rho(k, r) = F_\ell(k, r) , \quad \lim_{\rho \rightarrow \infty} G_\ell^\rho(k, r) = G_\ell(k, r) , \quad (2.31)$$

for a fixed  $r$  are analytically derived using the regular and irregular Coulomb wave functions,  $F_\ell(k, r)$  and  $G_\ell(k, r)$ , and various Wronskians between these wave functions and the free wave functions. They are given by

$$F_\ell^\rho(k, r) = \begin{cases} \frac{|F_\ell(k)|}{|F_\ell^\rho(k)|} F_\ell(k, r) & r \leq \rho \\ u_\ell(kr) \cos \delta_\ell^\rho + v_\ell(kr) \sin \delta_\ell^\rho & r \geq \rho \end{cases} ,$$

$$G_\ell^\rho(k, r) = \begin{cases} \frac{|F_\ell^\rho(k)|}{|F_\ell(k)|} G_\ell(k, r) + \frac{1}{k} (-W[G_\ell(k, r), u_\ell(kr)]_\rho \sin \delta_\ell^\rho \\ \quad + W[G_\ell(k, r), v_\ell(kr)]_\rho \cos \delta_\ell^\rho) F_\ell(k, r) & r \leq \rho \\ v_\ell(kr) \cos \delta_\ell^\rho - u_\ell(kr) \sin \delta_\ell^\rho & r \geq \rho . \end{cases} \quad (2.32)$$

The screened Coulomb wave function  $\psi_\ell^\rho(k, r)$  also has a expression similar to Eq. (2.29):

$$\psi_\ell^{\rho(+)}(k, r) = \frac{1}{k} \Im f_\ell^\rho(k, r) + f_\ell^\rho f_\ell^\rho(k, r) , \quad (2.33)$$

where  $f_\ell^\rho = (1/k)e^{i\delta_\ell^\rho} \sin \delta_\ell^\rho$ . Since  $f_\ell^\rho(k, r) = \omega_\ell^{(+)}(kr) = v_\ell(kr) + iu_\ell(kr)$  for  $r \geq \rho$ , the asymptotic form of Eq. (2.33) is

$$\psi_\ell^{\rho(+)}(k, r) = \frac{1}{k} u_\ell(kr) + f_\ell^\rho \omega_\ell^{(+)}(kr) \quad \text{for } r \geq \rho . \quad (2.34)$$

We multiply Eq. (2.33) by  $e^{i\zeta^\rho(k)}$  and find

$$\begin{aligned} \psi_\ell^{\rho(+)}(k, r) e^{i\zeta^\rho(k)} &= \frac{1}{k} \Im \left\{ f_\ell^\rho(k, r) e^{-i\zeta^\rho(k)} \right\} \\ &\quad + \left[ e^{i\zeta^\rho(k)} f_\ell^\rho e^{i\zeta^\rho(k)} + f_\eta^\rho \right] f_\ell^\rho(k, r) e^{-i\zeta^\rho(k)} , \end{aligned} \quad (2.35)$$

where we have set

$$f_\eta^\rho \equiv \frac{1}{2ik} \left( e^{2i\zeta^\rho(k)} - 1 \right) = \frac{1}{2ik} \left( e^{2i\eta \log(2k\rho)} - 1 \right) . \quad (2.36)$$

Here, we take the limit  $\rho \rightarrow \infty$  and use Eq. (2.28). If we compare the resultant expression with Eq. (2.29), we find the correspondence

$$e^{i\zeta^\rho(k)} f_\ell^\rho e^{i\zeta^\rho(k)} \rightarrow f_\ell^C - f_\eta^\rho \quad \text{as } \rho \rightarrow \infty . \quad (2.37)$$

In fact, Eq. (2.37) diverges, but if we add up over all the partial waves, the second term of Eq. (2.37) does not contribute except for  $\theta = 0$  because  $f_\eta^\rho$  is  $\ell$ -independent. Thus, the scattering amplitude of the sharply cut-off Coulomb force

$$f^\rho(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell^\rho P_\ell(\theta) , \quad (2.38)$$

satisfies

$$\lim_{\rho \rightarrow \infty} e^{i\zeta^\rho(k)} f^\rho(\theta) e^{i\zeta^\rho(k)} = f^C(\theta) \quad \text{for } \theta \neq 0 . \quad (2.39)$$

### §3. Two-body Coulomb problem

In this section, we consider a scattering problem for a two-body Coulomb system consisting of a short-range local potential  $v(r)$  with the interaction range  $a$  and the



Coulomb force  $\omega^C(r) = 2k\eta/r$ . The Schrödinger equation in the configuration space reads

$$\left[ \left( \frac{d}{dr} \right)^2 - \frac{\ell(\ell+1)}{r^2} - v(r) - \frac{2k\eta}{r} + k^2 \right] \Psi_\ell^{(+)}(r) = 0, \quad (3.1)$$

with the boundary condition

$$\Psi_\ell^{(+)}(r) \sim \frac{1}{k} \Im f_\ell(k, r) + f_\ell f_\ell(k, r) \quad (r \rightarrow \infty). \quad (3.2)$$

Here,  $f_\ell(k, r)$  is the Coulomb Jost solution in Eq. (A.3) and the partial-wave scattering amplitude  $f_\ell$  is expressed as

$$\begin{aligned} f_\ell &= f_\ell^C + e^{2i\sigma_\ell} f_\ell^N = \frac{1}{2ik} (e^{2i(\sigma_\ell + \delta_\ell^N)} - 1), \\ \text{with} \quad f_\ell^N &= \frac{1}{2ik} (e^{2i\delta_\ell^N} - 1), \end{aligned} \quad (3.3)$$

using the nuclear phase shift  $\delta_\ell^N$ . In the usual approach,  $\delta_\ell^N$  is calculated from the real regular function  $\mathcal{F}_\ell(k, r)$  for the Schrödinger equation Eq. (3.1), which satisfies the relationship

$$\Psi_\ell^{(+)}(r) = \frac{1}{k} e^{i(\sigma_\ell + \delta_\ell^N)} \mathcal{F}_\ell(k, r) \quad (3.4)$$

The asymptotic wave of  $\mathcal{F}_\ell(k, r)$  is expressed as

$$\begin{aligned} \mathcal{F}_\ell(k, r) &\sim F_\ell(k, r) \cos \delta_\ell^N + G_\ell(k, r) \sin \delta_\ell^N \\ &\sim \sin(kr - \eta \log 2kr - (\pi/2)\ell + \sigma_\ell + \delta_\ell^N) \quad (r \rightarrow \infty). \end{aligned} \quad (3.5)$$

The nuclear phase shift  $\delta_\ell^N$  is then calculated from Eq. (1.1) by assigning  $\mathcal{F}_\ell(k, r)$  to  $\psi_\ell$  and taking  $\rho > a$  large enough.

Similar equations are also valid for the sharply cut-off Coulomb force  $\omega_\rho(r)$  in Eq. (1.2). Namely, the Schrödinger equation for this system,

$$\left[ \left( \frac{d}{dr} \right)^2 - \frac{\ell(\ell+1)}{r^2} - v(r) - \omega_\rho(r) + k^2 \right] \Psi_\ell^{\rho(+)}(r) = 0, \quad (3.6)$$

has the asymptotic wave

$$\begin{aligned} \Psi_\ell^{\rho(+)}(r) &= \frac{1}{k} \Im f_\ell^\rho(k, r) + \bar{f}_\ell^\rho f_\ell^\rho(k, r) \quad \text{for } r \geq a \\ &= \frac{1}{k} u_\ell(kr) + \bar{f}_\ell^\rho \omega_\ell^{(+)}(k, r) \quad \text{for } r \geq \rho, \end{aligned} \quad (3.7)$$

where  $f_\ell^\rho(k, r)$  is the Jost solution for  $\omega_\rho(r)$ . The scattering amplitude  $\bar{f}_\ell^\rho$  in Eq. (3.7) this time is parametrized as

$$\begin{aligned} \bar{f}_\ell^\rho &= \frac{1}{2ik} (e^{2i\bar{\delta}_\ell^\rho} - 1) = f_\ell^\rho + e^{2i\bar{\delta}_\ell^\rho} f_\ell^{\rho N} \quad \text{with } \bar{\delta}_\ell^\rho = \delta_\ell^\rho + \delta_\ell^{\rho N}, \\ \text{and} \quad f_\ell^{\rho N} &= \frac{1}{2ik} (e^{2i\delta_\ell^{\rho N}} - 1), \end{aligned} \quad (3.8)$$

where  $f_\ell^\rho = (1/2ik)(e^{2i\delta_\ell^\rho} - 1)$  is the scattering amplitude for  $\omega_\rho(r)$ . Furthermore, we have the relationship

$$\begin{aligned}\Psi_\ell^{\rho(+)}(r) &= \frac{1}{k} e^{i\bar{\delta}_\ell^\rho} \mathcal{F}_\ell^\rho(k, r) , \\ \mathcal{F}_\ell^\rho(k, r) &= u_\ell(kr) \cos \bar{\delta}_\ell^\rho + v_\ell(kr) \sin \bar{\delta}_\ell^\rho \quad (r \geq \rho) .\end{aligned}\quad (3.9)$$

Note that the second equation of Eq. (3.9) is exact for the sharply cut-off Coulomb force. We multiply Eq. (3.7) by the phase factor  $e^{i\zeta^\rho(k)}$  and take the limit  $\rho \rightarrow \infty$ . Then, a procedure similar to Eq. (2.35) leads to the correspondence

$$\begin{aligned}\lim_{\rho \rightarrow \infty} \left[ e^{i\zeta^\rho(k)} \bar{f}_\ell^\rho e^{i\zeta^\rho(k)} + f_\eta^\rho \right] &= f_\ell^C + e^{2i\sigma_\ell} f_\ell^N , \\ \lim_{\rho \rightarrow \infty} \Psi_\ell^{\rho(+)}(k, r) e^{i\zeta^\rho(k)} &= \Psi_\ell^{(+)}(k, r) \quad \text{for } r \geq a .\end{aligned}\quad (3.10)$$

From Eq. (3.8), the nuclear phase shift  $\delta_\ell^N$  is obtained through

$$\delta_\ell^N = \lim_{\rho \rightarrow \infty} \delta_\ell^{\rho N} = \lim_{\rho \rightarrow \infty} (\bar{\delta}_\ell^\rho - \delta_\ell^\rho) . \quad (3.11)$$

The sharply cut-off Coulomb phase shift  $\delta_\ell^\rho$  is calculated from

$$\tan \delta_\ell^\rho = - \frac{W[F_\ell(k, r), u_\ell(k, r)]_\rho}{W[F_\ell(k, r), v_\ell(k, r)]_\rho} . \quad (3.12)$$

Since  $\bar{\delta}_\ell^\rho$  is obtained by solving the potential problem for  $v(r) + \omega_\rho(r)$ , Eqs. (3.11) and (3.12) gives a solution for the two-body Coulomb problem in the momentum representation, using the sharply cut-off Coulomb force.

Another method to derive the nuclear phase shift in the momentum representation is to use the two-potential formula for the  $T$ -matrix. For the short-range potential  $v$  and the sharply cut-off (or screened) Coulomb potential  $\omega_\rho$ , we solve the  $T$ -matrix equation

$$T^\rho = (v + \omega_\rho) + (v + \omega_\rho) G_0 T^\rho , \quad (3.13)$$

where  $G_0 = (z - h_0)^{-1}$  with  $z = E + i\varepsilon$  is the free Green function with the energy  $E$ . We assume the energy factor  $(\hbar^2/2\mu) = 1$  and set  $E = k^2$ . Furthermore, the partial wave decomposition is implicitly assumed and the orbital angular momentum  $\ell$  is omitted for typological simplicity. The kinetic energy operator  $h_0$  is, therefore,  $h_0 = (d/dr)^2 - \ell(\ell+1)/r^2$ . The two-potential formula for  $T^\rho$  is given by

$$\begin{aligned}T^\rho &= t_{\omega_\rho} + (1 + t_{\omega_\rho} G_0) \tilde{t}_{\omega_\rho} (1 + G_0 t_{\omega_\rho}) , \\ t_{\omega_\rho} &= \omega_\rho + \omega_\rho G_0 t_{\omega_\rho} = \omega_\rho + \omega_\rho G_{\omega_\rho} \omega_\rho \\ \tilde{t}_{\omega_\rho} &= v + v G_{\omega_\rho} \tilde{t}_{\omega_\rho} = v + v G^\rho v ,\end{aligned}\quad (3.14)$$

where  $G_{\omega_\rho} = (z - h_0 - \omega_\rho)^{-1}$  and  $G^\rho = (z - h_0 - v - \omega_\rho)^{-1}$ . To derive the scattering amplitude, we sandwich  $T^\rho$  with the plane wave (with the wave number  $k$ )

$$|\phi\rangle = \frac{1}{k} |u\rangle , \quad (3.15)$$

and define

$$|\psi^{\rho(+)}\rangle = |\phi\rangle + G_0\omega_\rho|\psi^{\rho(+)}\rangle. \quad (3.16)$$

Then, by using

$$|\psi^{\rho(+)}\rangle = (1 + G_0 t_{\omega_\rho})|\phi\rangle, \quad \langle\psi^{\rho(-)}| = \langle\phi|(1 + t_{\omega_\rho} G_0), \quad (3.17)$$

we find

$$\langle\phi|T^\rho|\phi\rangle = \langle\phi|t_{\omega_\rho}|\phi\rangle + \langle\psi^{\rho(-)}|\tilde{t}_{\omega_\rho}|\psi^{\rho(+)}\rangle. \quad (3.18)$$

In Eqs. (3.17) and (3.18),  $\langle\psi^{\rho(-)}|$  is defined by  $\psi^{(-)}(k, r) = (\psi^{(+)}(k, r))^*$ . This equation is essentially equivalent to the  $T$ -matrix in the distorted-wave Born approximation (DWBA). In fact, if we set

$$\tilde{t}_{\omega_\rho}|\psi^{\rho(+)}\rangle = v(1 + G_{\omega_\rho}\tilde{t}_{\omega_\rho})|\psi^{\rho(+)}\rangle \equiv v|\Psi^{\rho(+)}\rangle, \quad (3.19)$$

the on-shell  $T$ -matrix is expressed as

$$\langle\phi|T^\rho|\phi\rangle = \langle\phi|\omega_\rho|\psi^{\rho(+)}\rangle + \langle\psi^{\rho(-)}|v|\Psi^{\rho(+)}\rangle. \quad (3.20)$$

The LS equation for the total wave function

$$|\Psi^{\rho(+)}\rangle = |\psi^{\rho(+)}\rangle + G_{\omega_\rho}v|\Psi^{\rho(+)}\rangle \quad (3.21)$$

is equivalent to Eqs. (3.6) and (3.7).

Equation (3.18) gives a starting point for the “screening and renormalization procedure”. Namely, if we sandwich Eq. (3.18) with the renormalization phase  $e^{i\zeta^\rho}$  with  $\zeta^\rho = \zeta^\rho(k)$ , and take the limit  $\rho \rightarrow \infty$ , we find

$$\begin{aligned} \langle\phi|T|\phi\rangle &\equiv \lim_{\rho \rightarrow \infty} e^{i\zeta_\rho} \langle\phi|T^\rho|\phi\rangle e^{i\zeta_\rho} \\ &= \lim_{\rho \rightarrow \infty} e^{i\zeta_\rho} \langle\phi|t_\omega^\rho|\phi\rangle e^{i\zeta_\rho} + \lim_{\rho \rightarrow \infty} e^{i\zeta_\rho} \langle\psi^{\rho(-)}|\tilde{t}_\omega^\rho|\psi^{\rho(+)}\rangle e^{i\zeta_\rho} \\ &= \langle\phi|t_C|\phi\rangle + \lim_{\rho \rightarrow \infty} \langle\psi^{(-)}|\tilde{t}_\omega^\rho|\psi^{(+)}\rangle. \end{aligned} \quad (3.22)$$

Here,

$$|\psi^{(\pm)}\rangle = \lim_{\rho \rightarrow \infty} |\psi_\omega^{\rho(\pm)}\rangle e^{\pm i\zeta_\rho} \quad (3.23)$$

are the pure Coulomb wave functions. The first term in Eq. (3.22) is separated into the partial-wave Coulomb amplitude  $f_\ell^C$  and  $\ell$ -independent term from the discussion of the preceding section. When all the partial-wave contributions are added up, the first term becomes the pure Coulomb amplitude. Actually, the relationship between the scattering amplitude and the on-shell  $T$ -matrix yields

$$\langle\mathbf{q}_f|t_C|\mathbf{q}_i\rangle = -\frac{4\pi}{(2\pi)^3} \frac{\hbar^2}{2\mu} f^C(\theta) = -\frac{\hbar^2}{(2\pi)^2 \mu} f^C(\theta) \quad (3.24)$$

with  $|\mathbf{q}_f| = |\mathbf{q}_i| = k$ . In the second term of Eq. (3·22),  $\rho \rightarrow \infty$  limit can be taken, since the nuclear potential  $v$  is short-ranged. We define  $\tilde{t}$  by the solution of  $\tilde{t} = v + vG^C\tilde{t}$ , where  $G^C = (z - h_0 - 2\eta k/r)^{-1}$  is the Coulomb Green function. Thus, we find

$$\lim_{\rho \rightarrow \infty} \langle \psi^{(-)} | \tilde{t}_{\omega_\rho} | \psi^{(+)} \rangle = \langle \psi^{(-)} | \tilde{t} | \psi^{(+)} \rangle . \quad (3\cdot25)$$

To derive this matrix element, we introduce the total wave function  $|\Psi^{(+)}\rangle$  through

$$\tilde{t}|\psi^{(+)}\rangle = v(1 + G^C\tilde{t})|\psi^{(+)}\rangle = v|\Psi^{(+)}\rangle , \quad (3\cdot26)$$

which satisfies the LS equation

$$|\Psi^{(+)}\rangle = (1 + G^C\tilde{t})|\psi^{(+)}\rangle = |\psi^{(+)}\rangle + G^Cv|\Psi^{(+)}\rangle , \quad (3\cdot27)$$

and the Schrödinger equation in Eqs. (3·1) and (3·2). We should note that Eq. (3·27) has a solution, since  $v$  is short-ranged. Here, we introduce a decomposition of the partial-wave Green function

$$\begin{aligned} G^C(r, r'; k) &= -\psi_\ell(k, r_<)f_\ell(k, r_>) , \\ G^C &= \tilde{G}^C - |f_\ell\rangle\langle\psi_\ell^{(-)}| \quad \text{with} \quad \tilde{G}^C \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty . \end{aligned} \quad (3\cdot28)$$

Then, we find the asymptotic behavior

$$\begin{aligned} |\Psi^{(+)}\rangle &= |\psi^{(+)}\rangle - |f\rangle\langle\psi^{(-)}|v|\Psi^{(+)}\rangle + \tilde{G}^Cv|\Psi^{(+)}\rangle \\ &\sim |\psi^{(+)}\rangle - |f\rangle\langle\psi^{(-)}|\tilde{t}|\psi^{(+)}\rangle \quad \text{as} \quad r \rightarrow \infty . \end{aligned} \quad (3\cdot29)$$

If we use the Wronskians of the Coulomb wave functions

$$\begin{aligned} W[F_\ell, G_\ell] &= -k , \\ W[F_\ell, f_\ell] &= -ke^{-i\sigma_\ell} , \\ W[\psi_\ell^{(+)}, f_\ell] &= -1 , \end{aligned} \quad (3\cdot30)$$

derived from Eq. (A·9), we obtain

$$\begin{aligned} W[\psi_\ell^{(+)}, \Psi^{(+)}]_{r \rightarrow \infty} &= \langle \psi^{(-)} | \tilde{t} | \psi^{(+)} \rangle , \\ W[f_\ell, \Psi^{(+)}]_{r \rightarrow \infty} &= 1 \end{aligned} \quad (3\cdot31)$$

and

$$\langle \psi^{(-)} | \tilde{t} | \psi^{(+)} \rangle_\ell = \lim_{r \rightarrow \infty} \frac{W[\psi_\ell^{(+)}(k, r), \Psi^{(+)}(k, r)]}{W[f_\ell(k, r), \Psi^{(+)}(k, r)]} . \quad (3\cdot32)$$

If we further parametrize

$$\langle \psi^{(-)} | \tilde{t} | \psi^{(+)} \rangle_\ell = -e^{2i\sigma_\ell} \frac{1}{2ik} \left( e^{2i\delta_\ell^N} - 1 \right) , \quad (3\cdot33)$$

Eq. (3.32) is equivalent to

$$\tan \delta_\ell^N = - \lim_{r \rightarrow \infty} \frac{W[F_\ell(k, r), \Psi^{(+)}(k, r)]}{W[G_\ell(k, r), \Psi^{(+)}(k, r)]} . \quad (3.34)$$

Eventually, we find

$$\begin{aligned} \langle \phi | T | \phi \rangle &= - \frac{\hbar^2}{(2\pi)^2 \mu} f(\theta) , \\ f(\theta) &= f^C(\theta) + \sum_{\ell=0}^{\infty} (2\ell+1) e^{2i\sigma_\ell} f_\ell^N P_\ell(\cos \theta) , \\ \text{with } f_\ell^N &= \frac{1}{2ik} \left( e^{2i\delta_\ell^N} - 1 \right) . \end{aligned} \quad (3.35)$$

For practical calculations in the momentum space, it is much easier to start with the sharply cut-off Coulomb force from the very beginning. We multiply the LS equation in Eq. (3.21) by the renormalization phase  $e^{i\zeta_\rho}$  and take the limit  $\rho \rightarrow \infty$ . Then, by using Eqs. (2.11) and (2.21), we obtain

$$\lim_{\rho \rightarrow \infty} |\Psi^{\rho(+)}\rangle e^{i\zeta_\rho} = |\psi^{(+)}\rangle + G^C v \lim_{\rho \rightarrow \infty} |\Psi^{\rho(+)}\rangle e^{i\zeta_\rho} . \quad (3.36)$$

If we compare this with Eq. (3.27), we find

$$\lim_{\rho \rightarrow \infty} |\Psi^{\rho(+)}\rangle e^{i\zeta_\rho} = |\Psi^{(+)}\rangle . \quad (3.37)$$

If we further use Eq. (3.37) in  $\tan \delta_\ell^N$  of Eq. (3.34), we find

$$\tan \delta_\ell^N = - \lim_{r \rightarrow \infty} \frac{W[F_\ell(k, r), \Psi_\ell^{\rho(+)}(k, r)]}{W[G_\ell(k, r), \Psi_\ell^{\rho(+)}(k, r)]} , \quad (3.38)$$

for sufficiently large  $\rho$ . On the other hand, the asymptotic behavior of the wave functions for the short-range force yields

$$\begin{aligned} \Psi_\ell^{\rho(+)}(k, r) &= \frac{1}{k} e^{i\bar{\delta}_\ell^\rho} \left\{ u_\ell(kr) \cos \bar{\delta}_\ell^\rho + v_\ell(kr) \sin \bar{\delta}_\ell^\rho \right\} \\ &\quad \text{for } \rho < r \rightarrow \infty , \end{aligned} \quad (3.39)$$

for sufficiently large  $\rho$ . Thus, if we calculate Wronskians in Eq. (3.38) at  $r = \rho$ , we obtain

$$\begin{aligned} \tan \delta_\ell^N &= - \frac{W[F_\ell(k, r), \Psi_\ell^{\rho(+)}(k, r)]_{r=\rho}}{W[G_\ell(k, r), \Psi_\ell^{\rho(+)}(k, r)]_{r=\rho}} \\ &= - \frac{W[F_\ell, u_\ell]_\rho + \tan \bar{\delta}_\ell^\rho W[F_\ell, v_\ell]_\rho}{W[G_\ell, u_\ell]_\rho + \tan \bar{\delta}_\ell^\rho W[G_\ell, v_\ell]_\rho} , \end{aligned} \quad (3.40)$$

which is nothing but Eq. (1.3). After all, if  $\bar{\delta}_\ell^\rho$  is calculated in the momentum representation, the nuclear phase shift  $\delta_\ell^N$  is obtained through Eq. (3.40). The scattering amplitude  $f_\ell(\theta)$  is calculated from Eq. (3.35), using  $\delta_\ell^N$ .

#### §4. The screened Coulomb case

In this section, we will extend the preceding discussion for the sharply cut-off Coulomb force to a more general screened Coulomb force, which is formulated as

$$\omega_\rho(r) = \frac{2k\eta}{r} \alpha_\rho(r) , \quad (4.1)$$

according to Taylor.<sup>15)</sup> Here, the screening function  $\alpha_\rho(r)$  with  $1 \geq \alpha_\rho(r) \geq 0$  is a monotonically decreasing function of  $r$ , satisfying

- 1) with  $\rho$  fixed,  $\alpha_\rho(r)$  decreases to zero, faster than  $O(r^{-\varepsilon-2})$  ( $\varepsilon > 0$ ), as  $r$  approaches to  $\infty$ ,
- 2) with  $r$  fixed,  $\alpha_\rho(r)$  approaches to 1 as  $\rho$  approaches to  $\infty$ ,
- 3) around  $r \sim \rho$ , there exist sufficiently wide regions in which  $\alpha_\rho(r) \sim 1$  and  $\sim 0$ .

In 3) above, we added “an almost sharply cut-off condition” in addition to the original conditions 1) and 2) in Ref. 15). This condition is required if we wish to develop almost parallel discussion to the sharply cut-off Coulomb case, as seen below. Note that the sharply cut-off Coulomb case is included in the above category by taking  $\alpha_\rho(r) = \theta(\rho - r)$ .

The necessity to relax the sharply cut-off condition is as follows. First, in the LS-RGM formalism, the longest-range direct Coulomb potential becomes a screened Coulomb force as explicitly shown in §6.1 and §6.2. If the cluster wave functions are assumed to be standard harmonic-oscillator shell-model wave functions, the cut-off function  $\alpha_\rho(r)$  is usually expressed by the error function. Secondly, in the application to the  $pd$  elastic scattering, the asymptotic Hamiltonian involves a screened Coulomb force which is obtained from the  $pp$  Coulomb force by the folding procedure using a realistic deuteron wave function. In Ref. 7), the same  $pp$  screened Coulomb force is used for the  $pd$  screened Coulomb force, but using a more realistic  $pd$  Coulomb potential is certainly desirable to avoid unnecessary extra distortion of the deuteron in the asymptotic region by the Coulomb force. In any case, the screening function  $\alpha_\rho(r)$  should be chosen most appropriately for each problem, since “in practice Coulomb potentials are always screened” as stated in Ref. 15).

For the screened Coulomb force in Eq. (4.1), the parametrization of screened Coulomb wave functions in Eq. (2.30) is employed in the following, but the explicit solutions of  $F_\ell^\rho(k, r)$  and  $G_\ell^\rho(k, r)$  like in Eq. (2.32) are no longer available. In order to extend Eq. (2.32) to the screened Coulomb case, we first examine the behavior of the screened Coulomb wave functions around the origin  $r \sim 0$ . For the pure Coulomb solutions,  $F_\ell(k, r)$  and  $G_\ell(k, r)$ , we can easily show that

$$F_\ell(k, r) \sim \frac{1}{|F_\ell(k)|} u_\ell(kr) , \quad G_\ell(k, r) \sim |F_\ell(k)| v_\ell(kr) \quad \text{as } r \rightarrow 0 , \quad (4.2)$$

by using the explicit expression of the Coulomb Jost solution  $f_\ell(k, r)$  in Eq. (A.3) and the parametrization Eq. (A.9). The corresponding expressions for the screened Coulomb wave functions are

$$F_\ell^\rho(k, r) \sim \frac{1}{|F_\ell^\rho(k)|} u_\ell(kr)$$

$$G_\ell^\rho(k, r) \sim |F_\ell^\rho(k)| v_\ell(kr) + \frac{1}{|F_\ell^\rho(k)|} A_\ell^\rho(r) u_\ell(kr) \quad \text{as } r \rightarrow 0, \quad (4.3)$$

where an extra term including  $A_\ell^\rho(r)$  appears in the irregular solution  $G_\ell^\rho(k, r)$ . The real function  $A_\ell^\rho(r)$  is given by

$$\begin{aligned} A_\ell^\rho(r) &= -|F_\ell^\rho(k)| \frac{1}{k} W[G_\ell^\rho(k, r), v_\ell(kr)] \\ &\sim \begin{cases} \log r & \ell = 0 \\ \frac{1}{r^{2\ell}} & \ell \geq 1 \end{cases} \quad \text{as } r \rightarrow 0, \end{aligned} \quad (4.4)$$

and diverges as  $r \rightarrow 0$ . These results are derived by applying the Calogero's variable phase method<sup>24)</sup> to the regular solution  $\varphi_\ell^\rho(k, r)$  and the Jost solution  $f_\ell^\rho(k, r)$ .

For practical applications, we use the “almost sharply cut-off condition” 3) and assume a screening function satisfying

3)'

$$\alpha_\rho(r) = \begin{cases} 1 & \text{for } r < \rho - b = R_{\text{in}} \\ 0 & \text{for } r > \rho + b = R_{\text{out}} \end{cases}, \quad (4.5)$$

with a sufficiently large  $\rho \gg b$ . A new parameter  $b$  is introduced to make a smooth transition for the Coulomb force to disappear. To make the pure Coulomb region available,  $R_{\text{in}} \gg a$  should also be taken large enough, compared with the range  $a$  of the short-range nuclear force. By this assumption, we can extend the discussion in the sharply cut-off Coulomb case, although some modifications are necessary as seen below. First we apply Calogero's variable phase method to the regular function  $\varphi_\ell^\rho(k, r)$ . This solution and the pure Coulomb wave function  $\varphi_\ell(k, r)$  both satisfy the same integral equation Eq. (2.1) with  $\theta(\rho - r) \rightarrow \alpha_\rho(r)$  or 1 for  $r < R_{\text{in}}$ , yielding

$$\varphi_\ell^\rho(k, r) = \varphi_\ell(k, r) \quad \text{for } r \leq R_{\text{in}}. \quad (4.6)$$

If we use the standard relationship

$$\varphi_\ell(k, r) = \frac{1}{k^{\ell+1}} |F_\ell(k)| F_\ell(k, r), \quad (4.7)$$

(see Eqs. (A.9) and (2.30)), Eq. (4.6) implies

$$F_\ell^\rho(k, r) = \frac{|F_\ell(k)|}{|F_\ell^\rho(k)|} F_\ell(k, r) \quad \text{for } r \leq R_{\text{in}}. \quad (4.8)$$

Here, we can prove

$$\lim_{\rho \rightarrow \infty} \frac{|F_\ell(k)|}{|F_\ell^\rho(k)|} = 1. \quad (4.9)$$

Similarly, the screened Coulomb phase shift  $\delta_\ell^\rho = \delta_\ell^\rho(k)$  is proved to have the Coulomb limit

$$\lim_{\rho \rightarrow \infty} (\delta_\ell^\rho + \zeta^\rho) = \sigma_\ell, \quad (4.10)$$

where  $\zeta^\rho = \zeta^\rho(k)$  is given by Eq. (1.5) for  $\omega_\rho(r)$  in Eq. (4.1). For the irregular solution  $G_\ell^\rho(k, r)$ , the local phase approach does not work. In this case, we have admixture of the regular solution  $F_\ell(k, r)$  for  $r < R_{\text{in}}$ , which is related to  $A_\ell^\rho(r)$  in Eq. (4.4). Summarizing the above discussion, the explicit results of Eq. (2.32) in the sharply cut-off Coulomb case should be modified to

$$\begin{aligned} F_\ell^\rho(k, r) &= \frac{1}{a_\ell^\rho} F_\ell(k, r) , \\ G_\ell^\rho(k, r) &= a_\ell^\rho G_\ell(k, r) + A_\ell^\rho F_\ell(k, r) \quad \text{for } r \leq R_{\text{in}} , \\ F_\ell^\rho(k, r) &= u_\ell(kr) \cos \delta_\ell^\rho + v_\ell(kr) \sin \delta_\ell^\rho , \\ G_\ell^\rho(k, r) &= v_\ell(kr) \cos \delta_\ell^\rho - u_\ell(kr) \sin \delta_\ell^\rho \quad \text{for } r \geq R_{\text{out}} , \end{aligned} \tag{4.11}$$

where

$$a_\ell^\rho = \frac{|F_\ell^\rho(k)|}{|F_\ell(k)|} , \quad \lim_{\rho \rightarrow \infty} a_\ell^\rho = 1 , \tag{4.12}$$

and

$$\begin{aligned} \delta_\ell^\rho &\rightarrow \sigma_\ell - \zeta^\rho \quad \text{as } \rho \rightarrow \infty , \\ \text{with } \quad \zeta^\rho &= \zeta^\rho(k) = \frac{1}{2k} \int_{\frac{1}{2k}}^{\infty} \omega_\rho(r) dr . \end{aligned} \tag{4.13}$$

We note that, for the pure Coulomb problem, the renormalization of the screened Coulomb wave functions and the scattering amplitude is possible. In particular, Eqs. (2.33) - (2.39) are all valid owing to Eq. (4.13). However, the renormalization of the irregular solutions like in Eq. (2.19) needs a modification, since in general  $A_\ell^\rho \neq 0$  in Eq. (4.11). For example, the relationship in Eq. (2.28) should be modified as

$$\begin{aligned} \lim_{\rho \rightarrow \infty} \psi_\ell^{\rho(+)}(k, r) e^{i\zeta^\rho} &= \lim_{\rho \rightarrow \infty} \frac{1}{k} e^{i(\delta_\ell^\rho + \zeta^\rho)} F_\ell^\rho(k, r) = \frac{1}{k} e^{i\sigma_\ell} F_\ell(k, r) \\ &= \psi_\ell^{(+)}(k, r) \quad (\text{for regular Coulomb wave function}) , \\ \lim_{\rho \rightarrow \infty} f_\ell^\rho(k, r) e^{-i\zeta^\rho} &= \lim_{\rho \rightarrow \infty} e^{-i(\delta_\ell^\rho + \zeta^\rho)} [G_\ell^\rho(k, r) + iF_\ell^\rho(k, r)] \\ &= e^{-i\sigma_\ell} \left[ G_\ell(k, r) + \lim_{\rho \rightarrow \infty} A_\ell^\rho F_\ell(k, r) + iF_\ell(k, r) \right] \\ &= f_\ell(k, r) + A_\ell e^{-i\sigma_\ell} F_\ell(k, r) , \\ \lim_{\rho \rightarrow \infty} \left[ e^{i\zeta^\rho} f_\ell^\rho e^{i\zeta^\rho} + f_\eta^\rho \right] &= \frac{1}{2ik} (e^{i\sigma_\ell} - 1) = f_\ell^{\text{C}} , \end{aligned} \tag{4.14}$$

for  $r < R_{\text{in}}$ . Here, we have assumed that  $A_\ell = \lim_{\rho \rightarrow \infty} A_\ell^\rho$  exists for simplicity. By the same token, the  $\rho \rightarrow \infty$  limit in Eq. (2.35) becomes

$$\psi_\ell^{(+)}(k, r) = \frac{1}{k} \Im \{ f_\ell(k, r) + A_\ell e^{-i\sigma_\ell} F_\ell(k, r) \}$$



$$+f_\ell^C [f_\ell(k, r) + A_\ell e^{-i\sigma_\ell} F_\ell(k, r)] \quad \text{for } r < R_{\text{in}} . \quad (4.15)$$

Here, because  $A_\ell$  is real, the contribution from the terms proportional to  $A_\ell$  vanishes as

$$-\frac{1}{k} A_\ell \sin \sigma_\ell F_\ell(k, r) + \frac{1}{2ik} (e^{2i\sigma_\ell} - 1) A_\ell e^{-i\sigma_\ell} F_\ell(k, r) = 0 , \quad (4.16)$$

resulting in Eq. (2.29) again. It is important to note that this renormalization is possible only for the regular solution of the pure Coulomb problem. Once the nuclear potential is introduced, we need further renormalization for the magnitude of the wave function related to  $A_\ell^\rho$ , since the derivation of the regular solution also requires irregular solution of the screened Coulomb problem.

In order to make the similarity to the sharply cut-off Coulomb case more transparent, it is convenient to introduce a modified set of screened Coulomb wave functions by

$$\begin{aligned} \tilde{F}_\ell^\rho(k, r) &= a_\ell^\rho F_\ell^\rho(k, r) , \\ \tilde{G}_\ell^\rho(k, r) &= \frac{1}{a_\ell^\rho} G_\ell^\rho(k, r) - A_\ell^\rho F_\ell^\rho(k, r) . \end{aligned} \quad (4.17)$$

For  $r < R_{\text{in}}$ , these are the pure Coulomb wave functions:

$$\tilde{F}_\ell^\rho(k, r) = F_\ell(k, r) , \quad \tilde{G}_\ell^\rho(k, r) = G_\ell(k, r) \quad \text{for } r \leq R_{\text{in}} . \quad (4.18)$$

However, for  $r \geq R_{\text{out}}$ , Eq. (4.11) leads to

$$\begin{aligned} \tilde{F}_\ell^\rho(k, r) &= a_\ell^\rho [u_\ell(kr) \cos \delta_\ell^\rho + v_\ell(kr) \sin \delta_\ell^\rho] , \\ \tilde{G}_\ell^\rho(k, r) &= \frac{1}{a_\ell^\rho} [v_\ell(kr) \cos \delta_\ell^\rho - u_\ell(kr) \sin \delta_\ell^\rho] \\ &\quad - A_\ell^\rho [u_\ell(k, r) \cos \delta_\ell^\rho + v_\ell(k, r) \sin \delta_\ell^\rho] \quad \text{for } r \geq R_{\text{out}} . \end{aligned} \quad (4.19)$$

The Wronskians of these wave functions with the free scattering solutions in the asymptotic region are given by

$$\begin{aligned} \frac{1}{k} W [\tilde{F}_\ell^\rho(k, r), u_\ell(kr)] &= a_\ell^\rho \sin \delta_\ell^\rho , \\ \frac{1}{k} W [\tilde{F}_\ell^\rho(k, r), v_\ell(kr)] &= -a_\ell^\rho \cos \delta_\ell^\rho , \\ \frac{1}{k} W [\tilde{G}_\ell^\rho(k, r), u_\ell(kr)] &= \frac{1}{a_\ell^\rho} \cos \delta_\ell^\rho - A_\ell^\rho \sin \delta_\ell^\rho , \\ \frac{1}{k} W [\tilde{G}_\ell^\rho(k, r), v_\ell(kr)] &= \frac{1}{a_\ell^\rho} \sin \delta_\ell^\rho + A_\ell^\rho \cos \delta_\ell^\rho \quad \text{for } r \geq R_{\text{out}} . \end{aligned} \quad (4.20)$$

Let us assume  $a \ll R_{\text{in}}$  and consider the regular solution of the Schrödinger equation for  $v(r) + (2k\eta/r)\alpha_\rho(r)$ :

$$\Psi_\ell^{\rho(+)}(r) = \tilde{F}_\ell^\rho(k, r) \cos \delta_\ell^N + \tilde{G}_\ell^\rho(k, r) \sin \delta_\ell^N \quad \text{for } r > a . \quad (4.21)$$

In the  $a < r < R_{\text{in}}$  region,  $\delta_\ell^N$  becomes the nuclear phase shift owing to Eq. (4.18). This can be calculated from

$$\tan \delta_\ell^N = -\frac{W \left[ \tilde{F}_\ell^\rho(k, r), \Psi_\ell^{\rho(+)}(r) \right]}{W \left[ \tilde{G}_\ell^\rho(k, r), \Psi_\ell^{\rho(+)}(r) \right]} \quad \text{for } r > a. \quad (4.22)$$

The Wronskians in Eq. (4.22) can be calculated at any points  $r > a$ , since  $\tilde{F}_\ell^\rho(k, r)$ ,  $\tilde{G}_\ell^\rho(k, r)$  and  $\Psi_\ell^\rho(r)$  are all solutions of the Schrödinger equation for the screened Coulomb potential. In particular, the asymptotic behavior

$$\Psi_\ell^{\rho(+)}(r) = B \left[ u_\ell(kr) \cos \bar{\delta}_\ell^\rho + v_\ell(kr) \sin \bar{\delta}_\ell^\rho \right] \quad \text{for } r > R_{\text{out}} \quad (4.23)$$

without the Coulomb force, yields a connection condition

$$\tan \delta_\ell^N = -\frac{W \left[ \tilde{F}_\ell^\rho(k, r), u_\ell(kr) \right]_{R_{\text{out}}} + \tan \bar{\delta}_\ell^\rho W \left[ \tilde{F}_\ell^\rho(k, r), v_\ell(kr) \right]_{R_{\text{out}}}}{W \left[ \tilde{G}_\ell^\rho(k, r), u_\ell(kr) \right]_{R_{\text{out}}} + \tan \bar{\delta}_\ell^\rho W \left[ \tilde{G}_\ell^\rho(k, r), v_\ell(kr) \right]_{R_{\text{out}}}}, \quad (4.24)$$

which is an extension of Eq. (3.40) in the sharply cut-off Coulomb case. The phase shift  $\bar{\delta}_\ell^\rho$  is calculated from the standard procedure to solve  $T$ -matrix of  $v(r) + (2k\eta/r)\alpha_\rho(r)$  in the momentum representation.

To the contrary, we can also recover the asymptotic behavior of  $\Psi_\ell^{\rho(+)}(r)$  in Eq. (4.23), starting from Eq. (4.21) and Eq. (4.24). If we use the expressions of Wronskians in Eq. (4.20), the connection condition Eq. (4.24) yields

$$\tan \delta_\ell^N = \frac{a_\ell^\rho \sin \left( \bar{\delta}_\ell^\rho - \delta_\ell^\rho \right)}{\left[ \frac{1}{a_\ell^\rho} \cos \left( \bar{\delta}_\ell^\rho - \delta_\ell^\rho \right) + A_\ell^\rho \sin \left( \bar{\delta}_\ell^\rho - \delta_\ell^\rho \right) \right]}. \quad (4.25)$$

We write this as

$$\begin{aligned} \sin \delta_\ell^N &= \frac{a_\ell^\rho}{B_\ell^\rho} \sin \left( \bar{\delta}_\ell^\rho - \delta_\ell^\rho \right), \\ \cos \delta_\ell^N &= \frac{1}{B_\ell^\rho} \left[ \frac{1}{a_\ell^\rho} \cos \left( \bar{\delta}_\ell^\rho - \delta_\ell^\rho \right) + A_\ell^\rho \sin \left( \bar{\delta}_\ell^\rho - \delta_\ell^\rho \right) \right], \\ B_\ell^\rho &= \left\{ \left[ \frac{1}{a_\ell^\rho} \cos \left( \bar{\delta}_\ell^\rho - \delta_\ell^\rho \right) + A_\ell^\rho \sin \left( \bar{\delta}_\ell^\rho - \delta_\ell^\rho \right) \right]^2 + \left[ a_\ell^\rho \sin \left( \bar{\delta}_\ell^\rho - \delta_\ell^\rho \right) \right]^2 \right\}^{\frac{1}{2}}. \end{aligned} \quad (4.26)$$

If we use this in Eq. (4.21) for  $r > R_{\text{out}}$ , the asymptotic behavior of  $\tilde{F}_\ell^\rho(k, r)$  and  $\tilde{G}_\ell^\rho(k, r)$  in Eq. (4.19) yields Eq. (4.23) with  $B = 1/B_\ell^\rho$ . In particular, if  $v(r) = 0$ ,  $\bar{\delta}_\ell^\rho = \delta_\ell^\rho$  in Eq. (4.25) yields the correct results  $\delta_\ell^N = 0$ .

In fact,  $\delta_\ell^N$  in Eq. (4.21) is  $\rho$ -dependent:  $\delta_\ell^N = \delta_\ell^{\rho N}$ , and we need to take the limit  $\delta_\ell^N = \lim_{\rho \rightarrow \infty} \delta_\ell^{\rho N}$ . Furthermore, the present assumption that  $\alpha_\rho(r) = 1$  or

0 except for the interval  $[R_{\text{in}}, R_{\text{out}}] = [\rho - b, \rho + b]$ , is just an approximation. We have to examine the accuracy of this approximation for the finite  $\rho$  on the case-by-case basis. In practical calculations, we solve  $\tilde{F}_\ell^\rho(k, r)$  and  $\tilde{G}_\ell^\rho(k, r)$  from  $R_{\text{in}}$  to  $R_{\text{out}}$ , by taking the starting values of the pure Coulomb wave functions  $F_\ell(k, R_{\text{in}})$  and  $G_\ell(k, R_{\text{in}})$ . The Wronskians needed in Eq. (4.24) are calculated numerically. In the sharply cut-off Coulomb case with  $b = 0$  and  $R_{\text{in}} = R_{\text{out}} = \rho$ , this process is unnecessary, and reduced to Eq. (3.40).

The extra term proportional to  $A_\ell^\rho$  in Eq. (4.17) also affects the relationship of the Green function in Eq. (2.21). To find a new relationship for the screened Coulomb force, we solve Eq. (4.17) inversely and express  $F_\ell^\rho(k, r)$  and  $G_\ell^\rho(k, r)$  as

$$\begin{aligned} F_\ell^\rho(k, r) &= \frac{1}{a_\ell^\rho} \tilde{F}_\ell^\rho(k, r) , \\ G_\ell^\rho(k, r) &= a_\ell^\rho \tilde{G}_\ell^\rho(k, r) + A_\ell^\rho \tilde{F}_\ell^\rho(k, r) . \end{aligned} \quad (4.27)$$

Then the Green function of the screened Coulomb force in Eq. (2.20) is expressed for a fixed  $\ell$  as

$$G_\omega^\rho = \tilde{G}_\omega^\rho - \frac{1}{k} \frac{1}{a_\ell^\rho} A_\ell^\rho |\tilde{F}_\ell^\rho\rangle \langle \tilde{F}_\ell^\rho| , \quad (4.28)$$

with

$$\tilde{G}_\omega^\rho(r, r'; k) = -\frac{1}{k} \frac{1}{a_\ell^\rho} \tilde{F}_\ell^\rho(k, r_<) \left[ a_\ell^\rho \tilde{G}_\ell^\rho(k, r_>) + i \frac{1}{a_\ell^\rho} \tilde{F}_\ell^\rho(k, r_>) \right] . \quad (4.29)$$

For  $r, r' < R_{\text{in}}$ , the  $\rho \rightarrow \infty$  limit of Eq. (4.29) yields

$$\lim_{\rho \rightarrow \infty} \tilde{G}_\omega^\rho(r, r'; k) = G_\ell^C(r, r'; k) - \frac{1}{k} A_\ell |F_\ell\rangle \langle F_\ell| \quad \text{for } r, r' < R_{\text{in}} \rightarrow \infty . \quad (4.30)$$

We keep the finite  $\rho$  and write Eq. (3.21) as

$$\begin{aligned} |\Psi^{\rho(+)}\rangle &= |\psi^{\rho(+)}\rangle + \tilde{G}_\omega^\rho v |\Psi^{\rho(+)}\rangle - \frac{1}{k} \frac{1}{a_\ell^\rho} A_\ell^\rho |\tilde{F}_\ell^\rho\rangle \langle \tilde{F}_\ell^\rho| v |\Psi^{\rho(+)}\rangle \\ &= \frac{1}{k} e^{i\delta_\ell^\rho} \frac{1}{a_\ell^\rho} |\tilde{F}_\ell^\rho\rangle \left[ 1 - e^{-i\delta_\ell^\rho} A_\ell^\rho \langle \tilde{F}_\ell^\rho| v |\Psi^{\rho(+)}\rangle \right] + \tilde{G}_\omega^\rho v |\Psi^{\rho(+)}\rangle . \end{aligned} \quad (4.31)$$

Here, we define

$$|\Psi^{\rho(+)}\rangle = |\tilde{\Psi}^{\rho(+)}\rangle \left[ 1 - e^{-i\delta_\ell^\rho} A_\ell^\rho \langle \tilde{F}_\ell^\rho| v |\Psi^{\rho(+)}\rangle \right] . \quad (4.32)$$

Then, we find

$$|\tilde{\Psi}^{\rho(+)}\rangle = \frac{1}{k} e^{i\delta_\ell^\rho} \frac{1}{a_\ell^\rho} |\tilde{F}_\ell^\rho\rangle + \tilde{G}_\omega^\rho v |\tilde{\Psi}^{\rho(+)}\rangle . \quad (4.33)$$

Here, we multiply Eq. (4.33) by  $e^{i\zeta^\rho}$  and take a limit  $\rho \rightarrow \infty$  with  $r \in [a, R_{\text{in}}]$  fixed. The first term of the right-hand side of Eq. (4.33) is  $(1/k)e^{i\sigma_\ell} |F_\ell\rangle = |\psi_\ell^{(+)}\rangle$ . In the

second term, we further use the decomposition of the Green function

$$\begin{aligned}\tilde{G}_\omega^\rho &= \tilde{g}_{\omega_\rho}^{\text{res}} - \frac{1}{k} \frac{1}{a_\ell^\rho} \left[ a_\ell^\rho \tilde{G}_\ell^\rho + i(1/a_\ell^\rho) \tilde{F}_\ell^\rho \right] \langle \tilde{F}_\ell^\rho | \\ \tilde{g}_{\omega_\rho}^{\text{res}}(r, r'; k) &= -\frac{1}{k} \left\{ \tilde{F}_\ell^\rho(k, r) \tilde{G}_\ell^\rho(k, r') - \tilde{G}_\ell^\rho(k, r) \tilde{F}_\ell^\rho(k, r') \right\} \theta(r' - r),\end{aligned}\quad (4.34)$$

and find

$$\begin{aligned}\int_0^\infty dr' \tilde{G}_\omega^\rho(r, r'; k) v(r') \tilde{\Psi}^{\rho(+)}(r') e^{i\zeta^\rho} &= \int_r^\infty dr' \tilde{g}_{\omega_\rho}^{\text{res}}(r, r'; k) v(r') \tilde{\Psi}^{\rho(+)}(r') e^{i\zeta^\rho} \\ &- \frac{1}{k} \frac{1}{a_\ell^\rho} \left[ a_\ell^\rho \tilde{G}_\ell^\rho(k, r) + i \frac{1}{a_\ell^\rho} \tilde{F}_\ell^\rho(k, r) \right] \int_0^\infty dr' \tilde{F}_\ell^\rho(k, r') v(r') \tilde{\Psi}^{\rho(+)}(r') e^{i\zeta^\rho}.\end{aligned}\quad (4.35)$$

Here, the first integral in the right-hand side vanishes since  $v(r') = 0$  for  $r' > r > a$ . In the second integral, the range of  $v(r)$  makes  $r' < a$  only, so that we can safely replace  $\tilde{F}_\ell^\rho(k, r')$  by  $F_\ell(k, r')$ . Thus, we find

$$\begin{aligned}\tilde{G}_\omega^\rho v |\tilde{\Psi}^{\rho(+)}\rangle e^{i\zeta^\rho} &\sim -\frac{1}{k} [G_\ell + iF_\ell] \langle F_\ell | v |\tilde{\Psi}^{\rho(+)}\rangle e^{i\zeta^\rho} \\ &= -\frac{1}{k} e^{i\sigma_\ell} |f_\ell\rangle \langle F_\ell | v |\tilde{\Psi}^{\rho(+)}\rangle e^{i\zeta^\rho} \\ &= -|f_\ell\rangle \langle \psi^{(-)} | v |\tilde{\Psi}^{\rho(+)}\rangle e^{i\zeta^\rho} \quad \text{for } \rho \rightarrow \infty,\end{aligned}\quad (4.36)$$

and

$$\begin{aligned}\lim_{\rho \rightarrow \infty} |\tilde{\Psi}^{\rho(+)}\rangle e^{i\zeta^\rho} &= |\psi_\ell^{(+)}\rangle - |f_\ell\rangle \lim_{\rho \rightarrow \infty} \langle \psi^{(-)} | v |\tilde{\Psi}^{\rho(+)}\rangle e^{i\zeta^\rho} \\ &\quad \text{for } a < r < R_{\text{in}}.\end{aligned}\quad (4.37)$$

If we compare Eq. (4.37) with the asymptotic form in the exact Coulomb case in Eq. (3.29), we find

$$\lim_{\rho \rightarrow \infty} |\tilde{\Psi}^{\rho(+)}\rangle e^{i\zeta^\rho} = |\Psi^{(+)}\rangle. \quad (4.38)$$

In the matrix element of Eq. (4.32), we can also replace  $\tilde{F}_\ell^\rho$  by  $F_\ell$ , since  $v(r)$  is short-ranged. By solving Eq. (4.32) inversely, we can show that

$$|\Psi^{\rho(+)}\rangle = |\tilde{\Psi}^{\rho(+)}\rangle \left[ 1 + e^{-i\delta_\ell^\rho} A_\ell^\rho \langle F_\ell | v |\tilde{\Psi}^{\rho(+)}\rangle \right]^{-1}. \quad (4.39)$$

If we multiply Eq. (4.39) by  $e^{i\zeta^\rho}$  and take the limit  $\rho \rightarrow \infty$ ,  $\delta^\rho \rightarrow \sigma_\ell - \zeta^\rho$  yields

$$\lim_{\rho \rightarrow \infty} |\Psi^{\rho(+)}\rangle e^{i\zeta^\rho} = |\Psi^{(+)}\rangle \left[ 1 + e^{-i\sigma_\ell} A_\ell \langle F_\ell | v |\Psi^{(+)}\rangle \right]^{-1} \quad (4.40)$$

and

$$|\Psi^{(+)}\rangle = \lim_{\rho \rightarrow \infty} |\Psi^{\rho(+)}\rangle e^{i\zeta^\rho} \left[ 1 - e^{-i\sigma_\ell} A_\ell^\rho \langle F_\ell | v |\Psi^{\rho(+)}\rangle e^{i\zeta^\rho} \right]^{-1}. \quad (4.41)$$

This expression implies Eq. (3.37) is no longer valid for the screened Coulomb force, and we need an extra normalization factor  $[1 - e^{-i\sigma_\ell} A_\ell^\rho \langle F_\ell | v | \Psi^{\rho(+)} \rangle e^{i\zeta^\rho}]^{-1}$ .

Finally, we will show that another type of the connection condition, equivalent to Eq. (4.24), is also obtained by considering two types of asymptotic forms of  $|\tilde{\Psi}^{\rho(+)}\rangle$ . First, the asymptotic form of  $|\tilde{\Psi}^{\rho(+)}\rangle$  for  $R_{\text{in}} > r \rightarrow \infty$  is from Eqs. (4.33) and (4.35)

$$\begin{aligned} |\tilde{\Psi}^{\rho(+)}\rangle &\sim \frac{1}{k} e^{i\delta_\ell^\rho} \frac{1}{a_\ell^\rho} |\tilde{F}_\ell^\rho\rangle \\ &- \frac{1}{k} \frac{1}{a_\ell^\rho} \left[ a_\ell^\rho \tilde{G}_\ell^\rho(k, r) + i \frac{1}{a_\ell^\rho} \tilde{F}_\ell^\rho(k, r) \right] \langle F_\ell | v | \tilde{\Psi}^{\rho(+)}\rangle \quad \text{for } r \leq R_{\text{in}}. \end{aligned} \quad (4.42)$$

The Wronskians at  $r \rightarrow \infty$  with  $r \leq R_{\text{in}}$  are given by

$$\begin{aligned} W \left[ \tilde{F}_\ell^\rho, \tilde{\Psi}^{\rho(+)} \right] &= \langle F_\ell | v | \tilde{\Psi}^{\rho(+)}\rangle = e^{i\delta_\ell^\rho} \frac{1}{a_\ell^\rho} \langle F_\ell | v | \tilde{\Psi}_\ell^{\rho(+)}\rangle a_\ell^\rho e^{-i\delta_\ell^\rho}, \\ W \left[ \tilde{G}_\ell^\rho, \tilde{\Psi}^{\rho(+)} \right] &= e^{i\delta_\ell^\rho} \frac{1}{a_\ell^\rho} - i \frac{1}{(a_\ell^\rho)^2} \langle F_\ell | v | \tilde{\Psi}^{\rho(+)}\rangle \\ &= e^{i\delta_\ell^\rho} \frac{1}{a_\ell^\rho} \left\{ 1 - i \langle F_\ell | v | \tilde{\Psi}^{\rho(+)}\rangle \frac{1}{a_\ell^\rho} e^{-i\delta_\ell^\rho} \right\}. \end{aligned} \quad (4.43)$$

Thus, if we define  $\tilde{K}_\ell^\rho$  by

$$\tilde{K}_\ell^\rho \frac{1}{k} \langle F_\ell | v | \tilde{\Psi}^{\rho(+)}\rangle a_\ell^\rho e^{-i\delta_\ell^\rho} = 1 - i \langle F_\ell | v | \tilde{\Psi}^{\rho(+)}\rangle \frac{1}{a_\ell^\rho} e^{-i\delta_\ell^\rho}, \quad (4.44)$$

we obtain

$$\tilde{K}_\ell^\rho W \left[ \tilde{F}_\ell^\rho, \tilde{\Psi}^{\rho(+)} \right] = k W \left[ \tilde{G}_\ell^\rho, \tilde{\Psi}^{\rho(+)} \right]. \quad (4.45)$$

Here, we note that all the wave functions with tilde satisfy the Schrödinger equation for the screened Coulomb potential for  $r > a$ , so that we can evaluate Wronskians at any points  $r > a$ . If we take the limit  $\rho \rightarrow \infty$  in Eq. (4.44), Eq. (4.38) yields

$$\begin{aligned} \lim_{\rho \rightarrow \infty} \frac{1}{k} \langle F_\ell | v | \tilde{\Psi}^{\rho(+)}\rangle a_\ell^\rho e^{-i\delta_\ell^\rho} &= \frac{1}{k} \langle F_\ell | v | \Psi^{(+)}\rangle e^{-i\sigma_\ell} = \frac{1}{k} \langle F_\ell | \tilde{t}_\ell | \psi_\ell^{(+)}\rangle e^{-i\sigma_\ell} \\ &= \frac{1}{k^2} \langle F_\ell | \tilde{t}_\ell | F_\ell \rangle = e^{-2i\sigma_\ell} \langle \psi_\ell^{(-)} | \tilde{t}_\ell | \psi_\ell^{(+)}\rangle. \end{aligned} \quad (4.46)$$

Thus, if we define  $\lim_{\rho \rightarrow \infty} \tilde{K}_\ell^\rho = K_\ell^N$ , Eq. (4.44) becomes

$$K_\ell^N e^{-2i\sigma_\ell} \langle \psi_\ell^{(-)} | \tilde{t}_\ell | \psi_\ell^{(+)}\rangle = 1 - i e^{-2i\sigma_\ell} k \langle \psi_\ell^{(-)} | \tilde{t}_\ell | \psi_\ell^{(+)}\rangle. \quad (4.47)$$

If we further parametrize

$$\langle \psi_\ell^{(-)} | \tilde{t}_\ell | \psi_\ell^{(+)}\rangle = -e^{2i\sigma_\ell} \frac{1}{2ik} \left( e^{2i\delta_\ell^N} - 1 \right), \quad (4.48)$$

we find  $K_\ell^N = -k \cot \delta_\ell^N$ .

On the other hand, in the region  $r \geq R_{\text{out}}$ , the Coulomb-free asymptotic wave gives

$$|\Psi^{\rho(+)}\rangle = \frac{1}{k}|u_\ell\rangle - |\omega_\ell^{(+)}\rangle \langle\phi_\ell|T_\ell^\rho|\phi_\ell\rangle \quad \text{for } r \geq R_{\text{out}} , \quad (4.49)$$

where the  $T$ -matrix  $T_\ell^\rho$  is defined in Eq. (3.13). If we write Eq. (4.49) by the  $K$ -matrix defined by

$$K_\ell^\rho \langle\phi_\ell|T_\ell^\rho|\phi_\ell\rangle = 1 - ik \langle\phi_\ell|T_\ell^\rho|\phi_\ell\rangle , \quad (4.50)$$

it is expressed as

$$|\Psi^{\rho(+)}\rangle = [|u_\ell\rangle K_\ell^\rho - |v_\ell\rangle k] \frac{1}{k} \langle\phi_\ell|T_\ell^\rho|\phi_\ell\rangle \quad \text{for } r \geq R_{\text{out}} . \quad (4.51)$$

We can use this to calculate the Wronskians in Eq. (4.45) at  $r = R_{\text{out}}$ , since the difference between  $\tilde{\Psi}_\ell^{\rho(+)}$  and  $\Psi_\ell^{\rho(+)}$  is just a normalization. From these processes, we eventually obtain

$$\tilde{K}_\ell^\rho \left\{ W[\tilde{F}_\ell^\rho, u_\ell]_{R_{\text{out}}} K_\ell^\rho - kW[\tilde{F}_\ell^\rho, v_\ell]_{R_{\text{out}}} \right\} = k \left\{ W[\tilde{G}_\ell^\rho, u_\ell]_{R_{\text{out}}} K_\ell^\rho - kW[\tilde{G}_\ell^\rho, v_\ell]_{R_{\text{out}}} \right\} , \quad (4.52)$$

which is equivalent to Eq. (4.24) since  $K_\ell^\rho = -k \cot \bar{\delta}_\ell^\rho$  and  $\tilde{K}_\ell^\rho = -k \cot \delta_\ell^{\rho N}$ .

Summarizing this section, we first calculate  $\langle\phi_\ell|T_\ell^\rho|\phi_\ell\rangle$  from the LS equation in the momentum representation, calculate  $K_\ell^\rho$  by Eq. (4.50), transform to  $\tilde{K}_\ell^\rho$  by Eq. (4.52), and take the limit  $\lim_{\rho \rightarrow \infty} \tilde{K}_\ell^\rho = K_\ell^N$ . Then, the nuclear phase shift  $\delta_\ell^N$  is obtained from  $K_\ell^N = -k \cot \delta_\ell^N$ .

## §5. Application to the $pd$ scattering

Application of the present formalism to the  $pd$  scattering is not straightforward because of several reasons. First, the asymptotic  $pd$  Coulomb potential suffers the strong distortion effect of the deuteron due to the long-range nature of the Coulomb force. In the strict three-body treatment of the  $nd$  scattering by the AGS equation, the distortion effect of the deuteron is fully taken into account, but only for the short-range force. Even if we neglect the Coulomb distortion effect by using the screened Coulomb force, the quasi-singular nature of this interaction causes the difficulty that the treatment by the standard AGS equation eventually breaks down at the limit of  $\rho \rightarrow \infty$ . To avoid this, a new formulation by the Coulomb-modified AGS equation was devised. However, very singular behavior of the screened Coulomb wave functions in the momentum representation makes it difficult to solve this equation numerically. Another difficulty lies in the partial-wave expansion of the AGS equation. Even in the two-body Coulomb problem, the partial-wave expansion of the Coulomb amplitude does not converge in the usual sense, but converges only as the distribution. It is therefore attempted to formulate the AGS equation based on the three-dimensional description of the two-body  $t$ -matrix.<sup>1),2),3),4),5)</sup> The isospin

symmetry breaking by the  $T = 3/2$  component should also be taken into account, since the Coulomb force admixes the different isospins. Here, we extend the “screening and renormalization technique” to incorporate the present approach and try to find a practical method to deal with the  $pd$  elastic scattering even in an approximate way.

Let  $\omega^\rho(\mathbf{r}; 1, 2)$  be a screened Coulomb force acting between two nucleons 1 and 2:

$$\omega^\rho(\mathbf{r}; 1, 2) = \frac{e^2}{r} \alpha_\rho(r) \frac{1 + \tau_z(1)}{2} \frac{1 + \tau_z(2)}{2} . \quad (5.1)$$

Here,  $\mathbf{r}$  is the relative coordinate between the two nucleons. We use a set of Jacobi coordinates of particles (1-2)+3 as the standard one and denote it by  $\gamma = 3$ . Another relative coordinate is denoted by  $\mathbf{R}$  in this section. Then, the screened Coulomb potential Eq. (5.1) is expressed as  $\omega_\gamma^\rho$  with  $\gamma = 3$ . In the following, we formulate the Coulomb-modified AGS equation in the isospin representation. The three-particle symmetric three-body screened Coulomb potential  $\omega_C^\rho = \sum_\alpha \omega_\alpha^\rho$  is given in the isospin basis as<sup>25)</sup>

$$\omega_C^\rho = \omega_\gamma^\rho + \mathcal{W}_\gamma^\rho + W_\gamma^\rho \quad \text{for } \forall \gamma . \quad (5.2)$$

Here,  $W_\gamma^\rho$  denotes the screened Coulomb potential between the nucleon  $\gamma$  and the residual  $NN$  pair, and is a function of the Jacobi coordinate  $\mathbf{R}_\gamma$  between them. Furthermore, the three-body potential  $\mathcal{W}_\gamma^\rho$ , which is usually called the polarization potential,<sup>25)</sup> is defined by

$$\mathcal{W}_\gamma^\rho = \sum_\beta \left( \bar{\delta}_{\gamma,\beta} \omega_\beta^\rho - \delta_{\gamma,\beta} W_\beta^\rho \right) = \sum_\beta \bar{\delta}_{\gamma,\beta} \omega_\beta^\rho - W_\gamma^\rho . \quad (5.3)$$

It should be noted that, for  $ppn$  system, either of  $\omega_\gamma^\rho$  or  $\mathcal{W}_\gamma^\rho + W_\gamma^\rho$  in Eq. (5.2) is only non-zero.

The two-potential formula for the three-body system is derived for the solutions of the Coulomb-modified AGS equation.<sup>7)</sup> First, the three-body transition operator  $U_{\beta,\alpha}^\rho$  for the usual AGS equation is defined through,

$$\mathcal{G}^\rho = \delta_{\beta,\alpha} g_\alpha^\rho + g_\beta^\rho U_{\beta,\alpha}^\rho g_\alpha^\rho , \quad (5.4)$$

where the full resolvent  $\mathcal{G}^\rho$  and the channel resolvent  $g_\alpha^\rho$  are defined by

$$\mathcal{G}^\rho = \left( z - H_0 - \sum_\alpha v_\alpha - \omega_C^\rho \right)^{-1} , \quad g_\alpha^\rho = (z - H_0 - v_\alpha - \omega_\alpha^\rho)^{-1} , \quad (5.5)$$

with  $v_\alpha$  being the short-range nuclear potential and  $z = E + \varepsilon_d + i0$  composed of the incident energy  $E$  and the deuteron energy  $\varepsilon_d$ . The three-body kinetic-energy operator is expressed as  $H_0 = h_{0\gamma} + \bar{h}_{0\gamma}$  for an arbitrary set of Jacobi coordinate  $\gamma$ . The transition operator  $U_{\beta,\alpha}^\rho$  satisfies the AGS equation

$$U_{\beta,\alpha}^\rho = \bar{\delta}_{\beta,\alpha} G_0^{-1} + \sum_\sigma \bar{\delta}_{\beta,\sigma} t_\sigma^\rho G_0 U_{\sigma,\alpha}^\rho , \quad (5.6)$$

where  $G_0^{-1} = (z - H_0)^{-1}$  is the free resolvent and basic two-nucleon  $T$ -matrix  $t_\sigma^\rho$  is generated by solving the LS equation for  $v_\sigma + \omega_\sigma^\rho$ . Namely,

$$t^\rho = (v + \omega^\rho) + (v + \omega^\rho) G_0 t^\rho . \quad (5.7)$$

In Eq. (5.6) and below, we use the usual convention  $\bar{\delta}_{\beta,\alpha} = 1 - \delta_{\beta,\alpha}$ . The full resolvent  $\mathcal{G}^\rho$  can also be decomposed as

$$\mathcal{G}^\rho = \delta_{\beta,\alpha} G_\alpha^\rho + G_\beta^\rho \tilde{U}_{\beta,\alpha}^\rho G_\alpha^\rho , \quad (5.8)$$

using another resolvent  $G_\alpha^\rho$  defined by

$$G_\alpha^\rho = (z - H_0 - v_\alpha - \omega_\alpha^\rho - W_\alpha^\rho)^{-1} , \quad (5.9)$$

The operator  $\tilde{U}_{\beta,\alpha}^\rho$  satisfies the Coulomb-modified AGS equation:<sup>7)</sup>

$$\tilde{U}_{\beta,\alpha}^\rho = \bar{\delta}_{\beta,\alpha} ((G_\alpha^\rho)^{-1} + v_\alpha) + \delta_{\beta,\alpha} \mathcal{W}_\alpha^\rho + \sum_\sigma (\bar{\delta}_{\beta,\sigma} v_\sigma + \delta_{\beta,\sigma} \mathcal{W}_\sigma^\rho) G_\sigma^\rho \tilde{U}_{\sigma,\alpha}^\rho . \quad (5.10)$$

From the relationship between  $g_\alpha^\rho$  and  $G_\alpha^\rho$ , the operator  $\tilde{U}_{\beta,\alpha}^\rho$  is related to  $U_{\beta,\alpha}^\rho$  through

$$U_{\beta,\alpha}^\rho = \delta_{\beta,\alpha} T_\alpha^\rho + (1 + T_\beta^\rho g_\beta^\rho) \tilde{U}_{\beta,\alpha}^\rho (1 + g_\alpha^\rho T_\alpha^\rho) , \quad (5.11)$$

where the screened Coulomb  $T$ -matrix  $T_\alpha^\rho$  for the  $pd$  scattering is obtained from  $W_\alpha^\rho$  through

$$T_\alpha^\rho = W_\alpha^\rho + W_\alpha^\rho g_\alpha^\rho T_\alpha^\rho = W_\alpha^\rho + W_\alpha^\rho G_\alpha^\rho W_\alpha^\rho . \quad (5.12)$$

Equation (5.11) is the two-potential formula for the three-body system. The Coulomb-distorted asymptotic wave function is defined by  $|\psi_\alpha^{\rho(+)}\rangle = (1 + g_\alpha^\rho T_\alpha^\rho) |\phi_\alpha\rangle$  from the channel wave function  $|\phi_\alpha\rangle = |\mathbf{q}_{0\alpha}, \psi_\alpha^d\rangle$ . From this definition and Eq. (5.12), we obtain

$$|\psi_\alpha^{\rho(+)}\rangle = |\phi_\alpha\rangle + g_\alpha^\rho W_\alpha^\rho |\psi_\alpha^{\rho(+)}\rangle . \quad (5.13)$$

We define  $|\psi_\alpha^{\rho(-)}\rangle$  as the complex conjugate of  $|\psi_\alpha^{\rho(+)}\rangle$  and find

$$\langle \phi_\beta | U_{\beta,\alpha}^\rho | \phi_\alpha \rangle = \delta_{\beta,\alpha} \langle \phi_\alpha | T_\alpha^\rho | \phi_\alpha \rangle + \langle \psi_\beta^{\rho(-)} | \tilde{U}_{\beta,\alpha}^\rho | \psi_\alpha^{\rho(+)} \rangle . \quad (5.14)$$

We can separate the deuteron part in Eq. (5.13) and we obtain

$$\begin{aligned} |\psi_\alpha^{\rho(+)}\rangle &= |\chi_\alpha^{\rho(+)}\rangle, \psi_\alpha^d\rangle , \\ |\chi_\alpha^{\rho(+)}\rangle &= |\mathbf{q}_{0\alpha}\rangle + (E_\alpha + i0 - \bar{h}_{0\alpha})^{-1} W_\alpha^\rho |\chi_\alpha^{\rho(+)}\rangle , \end{aligned} \quad (5.15)$$

where  $E_\alpha$  is the incident energy in the  $\alpha$ -channel and the deuteron wave function  $|\psi_\alpha^d\rangle$  satisfies

$$(\varepsilon_d - h_{0\alpha} - v_\alpha - \omega_\alpha^\rho) |\psi_\alpha^d\rangle = 0 . \quad (5.16)$$



Note that  $\omega^\rho$  does not actually contribute in Eq. (5.16), since the isospin of the deuteron is zero. From Eq. (5.15), we find

$$(G_\alpha^\rho)^{-1}|\psi_\alpha^{\rho(+)}\rangle = (E_\alpha - \bar{h}_{0\alpha} - W_\alpha^\rho)|\chi_\alpha^{\rho(+)}, \psi_\alpha^d\rangle = 0. \quad (5.17)$$

For three identical particles in the isospin formalism, a transition operator to the channel  $\gamma$ ,  $\tilde{U}_\gamma^\rho$ , is defined through

$$\sum_\alpha \tilde{U}_{\gamma,\alpha}^\rho |\psi_\alpha^{\rho(+)}\rangle \equiv \tilde{U}_\gamma^\rho |\psi_\gamma^{\rho(+)}\rangle. \quad (5.18)$$

We assume  $\gamma$  to be the standard coordinate system  $\gamma = 3$  and abbreviate the subscript  $\gamma$ . Then, we obtain from Eqs. (5.10) and (5.17) the Coulomb-modified AGS equation for three identical particles:

$$\tilde{U}^\rho |\psi^{\rho(+)}\rangle = (Pv + \mathcal{W}^\rho) |\psi^{\rho(+)}\rangle + (Pv + \mathcal{W}^\rho) G^\rho \tilde{U}^\rho |\psi^{\rho(+)}\rangle, \quad (5.19)$$

where  $G^\rho = (z - H_0 - v - \omega^\rho - W^\rho)^{-1}$  and  $P = P_{(12)}P_{(23)} + P_{(13)}P_{(23)}$  is the permutation operator for the rearrangement. In Eq. (5.19), we set  $\tilde{U}^\rho |\psi^{\rho(+)}\rangle = (Pv + \mathcal{W}^\rho) |\Psi^{\rho(+)}\rangle$  and obtain

$$|\Psi^{\rho(+)}\rangle = |\psi^{\rho(+)}\rangle + G^\rho (Pv + \mathcal{W}^\rho) |\Psi^{\rho(+)}\rangle. \quad (5.20)$$

Here,  $|\Psi^{\rho(+)}\rangle$  is the total wave function for the screened Coulomb problem and is related to the total wave function for the full Coulomb problem  $|\Psi^{(+)}\rangle$  through <sup>\*)</sup>

$$\lim_{\rho \rightarrow \infty} |\Psi^{\rho(+)}\rangle e^{i\zeta^\rho} = |\Psi^{(+)}\rangle \quad (5.21)$$

with a shift function  $\zeta^\rho$ . The shift function  $\zeta^\rho = \zeta^\rho(k)$  is defined by

$$\zeta^\rho(q_0) = \frac{1}{2q_0} \int_{\frac{1}{2q_0}}^\infty W^\rho(R) dR, \quad (5.22)$$

where  $q_0$  is the wave number between the incident proton and the deuteron in the center-of-mass (cm) system. The “screening and renormalization procedure”<sup>7)</sup> converts Eq. (5.14) to its full Coulomb correspondence

$$\langle \phi | U^C | \phi \rangle = \langle \phi | T^C | \phi \rangle + \langle \psi^{C(-)} | \tilde{U}^C | \psi^{C(+)} \rangle. \quad (5.23)$$

Equation (5.20) is the distorted-wave version of

$$|\Psi^{\rho(+)}\rangle = |\phi\rangle + g^\rho P(v + \omega^\rho) |\Psi^{\rho(+)}\rangle, \quad (5.24)$$

which can be derived similarly from the AGS equation in Eq. (5.6) by assigning  $U^\rho |\phi\rangle = P(v + \omega^\rho) |\Psi^{\rho(+)}\rangle$ . In fact, if we note that  $|\Psi^{\rho(+)}\rangle$  is three-nucleon anti-symmetric, we can easily derive Eq. (5.19) from Eq. (5.24) by using  $P(v + \omega^\rho) =$

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<sup>\*)</sup> Strictly speaking, this relationship is valid only for the sharply cutoff Coulomb potential. For general screened Coulomb potentials, an extra finite normalization factor like in Eq. (4.32) is necessary for  $|\Psi^{\rho(+)}\rangle$ . The following relations are all valid by modifying  $|\Psi^{\rho(+)}\rangle$  to  $|\tilde{\Psi}^{\rho(+)}\rangle$ .

$Pv + \mathcal{W}^\rho + W^\rho$ . On the other hand, the Faddeev components  $|\Psi^\rho\rangle$ , satisfying  $|\Psi^{\rho(+)}\rangle = (1 + P)|\Psi^\rho\rangle$ , can be derived by setting  $G_0 U^\rho |\phi\rangle = P|\Psi^\rho\rangle$  in the AGS equation:

$$|\Psi^\rho\rangle = |\phi\rangle + G_0 t^\rho P |\Psi^\rho\rangle. \quad (5.25)$$

In the isospin formalism for the total isospin  $T = 1/2$  state, we use the effective  $T$ -matrix  $t^{I=1} = (2/3)t_{pp}^\rho + (1/3)t_{np}$  for the isospin 1  $NN$  channel.<sup>26)</sup>

Instead of using the “screening and renormalization” procedure, we use an extension of Vincent and Phatak procedure<sup>6)</sup> of the two-cluster Coulomb problem, which is equivalent to the “screening and renormalization procedure” in the limit of  $\rho \rightarrow \infty$ . The scattering amplitude is obtained by imposing a connection condition on the  $K$ -matrix<sup>\*)</sup>  $K_{\alpha,\beta}^\rho \equiv (Z^{-1})_{\alpha,\beta} - \langle \phi_\alpha | X^\rho | \phi_\beta \rangle$  for the  $pd$  scattering,<sup>28)</sup> which is derived from the two different asymptotic forms of the total wave function in Eqs. (5.20) and (5.24). From here on, the subscripts  $\alpha, \beta$ , etc. specify the channel quantum numbers. We define a reduced wave function  $\Phi_{\alpha,\gamma}^{\rho(+)}(R) \equiv \langle R, \psi_\alpha^d | \Psi_\gamma^{\rho(+)} \rangle$ . The asymptotic form for the wave function Eq (5.24) is without a constant normalization factor

$$\Phi_{\alpha,\gamma}^{\rho(+)}(R) \sim u_\alpha(q_0 R) K_{\alpha,\gamma}^\rho - c v_\alpha(q_0 R) \delta_{\alpha,\gamma} \quad \text{for} \quad R > R_{\text{out}}, \quad (5.26)$$

where  $c = q_0(\pi/2)(4M_N/3\hbar^2)$  with  $M_N$  being the nucleon mass. For the total wave function Eq. (5.20), the asymptotic form is

$$\begin{aligned} \Phi_{\alpha,\gamma}^{\rho(+)}(R) &\sim \frac{1}{q_0} \sum_\beta \left\{ \tilde{F}_\alpha^\rho(q_0, R) \tilde{K}_{\alpha,\beta}^\rho - c \tilde{G}_\alpha^\rho(q_0, R) \delta_{\alpha,\beta} \right\} \\ &\times \frac{1}{q_0} \langle F_\beta, \psi_\beta^d | (Pv + \mathcal{W}^\rho) | \Psi_\gamma^{\rho(+)} \rangle \quad \text{for} \quad R > a, \end{aligned} \quad (5.27)$$

where  $a$  is the range of the nuclear force. Here,  $\tilde{F}_\alpha^\rho$  and  $\tilde{G}_\alpha^\rho$  are the screened Coulomb wave functions defined in Eq. (4.17). In the inside region  $R < R_{\text{in}}$ ,  $\tilde{F}_\alpha$  and  $\tilde{G}_\alpha$  are equal to  $F_\alpha$  and  $G_\alpha$ , respectively. The connection condition for  $\Phi_{\alpha,\gamma}^{\rho(+)}(R)$  at  $R = R_{\text{out}}$  is written in terms of Wronskians:

$$\begin{aligned} &\sum_\beta \tilde{K}_{\alpha,\beta}^\rho \left\{ W[\tilde{F}_\beta^\rho, u_\beta]_{R_{\text{out}}} K_{\beta,\gamma}^\rho - W[\tilde{F}_\beta^\rho, v_\beta]_{R_{\text{out}}} c \delta_{\beta,\gamma} \right\} \\ &= c \left\{ W[\tilde{G}_\alpha^\rho, u_\alpha]_{R_{\text{out}}} K_{\alpha,\gamma}^\rho - W[\tilde{G}_\alpha^\rho, v_\alpha]_{R_{\text{out}}} c \delta_{\alpha,\gamma} \right\}. \end{aligned} \quad (5.28)$$

Matrix elements  $\tilde{U}_{\beta,\gamma}^\rho$ , defined by

$$\sum_\beta \left[ \tilde{K}_{\alpha,\beta}^\rho + i c \delta_{\alpha,\beta} \right] \tilde{U}_{\beta,\gamma}^\rho = \delta_{\alpha,\gamma} \quad (5.29)$$

in the limit of  $\rho \rightarrow \infty$ , are related to  $\langle \psi_\beta^{\rho(-)} | \tilde{U}^\rho | \psi_\gamma^{\rho(+)} \rangle$  through

$$\langle \psi_\beta^{\rho(-)} | \tilde{U}^\rho | \psi_\gamma^{\rho(+)} \rangle = e^{i(\sigma_\beta + \sigma_\gamma)} \tilde{U}_{\beta,\gamma}^\rho. \quad (5.30)$$

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<sup>\*)</sup> Here, the  $K$ -matrix is defined by the form of  $K_\ell(k) = (1/k) \tan \delta_\ell(k)$  for the on-shell matrix elements.

Here,  $\sigma_\beta$  and  $\sigma_\gamma$  are the Coulomb phase shifts in the channels  $\beta$  and  $\gamma$ , respectively. The scattering amplitude  $f_{\beta,\gamma}^{N,\rho}$  is obtained from  $\tilde{U}_{\beta,\gamma}^\rho$  through

$$f_{\beta,\gamma}^{N,\rho} = -\frac{\pi}{2} \frac{4M_N}{3\hbar^2} \tilde{U}_{\beta,\gamma}^\rho . \quad (5.31)$$

In the channel-spin representation, the full scattering amplitude is written as

$$\begin{aligned} f_{S'_c S'_{cz}, S_c S_{cz}}^\rho(\hat{\mathbf{q}}_f, \hat{\mathbf{q}}_i) &= \delta_{S'_c, S_c} \delta_{S'_{cz}, S_{cz}} f^C(\theta) + 4\pi \sum_{\ell' \ell J J_z} e^{i(\sigma_{\ell'} + \sigma_\ell)} f_{(\ell' S'_c), (\ell S_c)}^{N J, \rho} \\ &\times \sum_{m'} \langle \ell' m' S'_c S'_{cz} | J J_z \rangle Y_{\ell' m'}(\hat{\mathbf{q}}_f) \sum_m \langle \ell m S_c S_{cz} | J J_z \rangle Y_{\ell m}^*(\hat{\mathbf{q}}_i) , \end{aligned} \quad (5.32)$$

for a sufficiently large  $\rho$ .

## §6. Numerical performance

### 6.1. Comparison with the exact solutions for the Ali-Bodmer $\alpha\alpha$ potential

Ali-Bodmer  $\alpha\alpha$  potential is a simple phenomenological potential which reproduces the results of the phase-shift analysis for the  $\alpha\alpha$  scattering up to  $E_{\text{cm}} \sim 15$  MeV. The angular-momentum-dependent version called Ali-Bodmer d (ABd) has the explicit form

$$V_{\alpha\alpha}^{\text{ABd}}(r) = V_1 e^{-\eta_1 r^2} + V_2 e^{-\eta_2 r^2} + \frac{4e^2}{r} \text{erf}(\beta r) , \quad (6.1)$$

with the parameters  $\eta_1 = 0.7^2 \text{ fm}^{-2}$ ,  $\eta_2 = 0.475^2 \text{ fm}^{-2}$ ,  $V_2 = -130 \text{ MeV}$  and

$$\begin{aligned} V_1 &= \begin{cases} 500 \text{ MeV} & \text{for } S \\ 320 \text{ MeV} & \text{for } D \\ 0 & \text{for } \ell \geq 4 \end{cases} , \\ \beta &= \frac{\sqrt{3}}{2 \times 1.44} = 0.6014 \dots \text{ fm}^{-1} . \end{aligned} \quad (6.2)$$

In Eq. (6.1),  $\text{erf}(x)$  stands for the error function defined by  $\text{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$ . Since this potential model is exactly solvable by the Runge-Kutta-Gill (RKG) method, it is suitable to test the accuracy of the Coulomb approach developed in this paper. With the assignment  $\alpha = 4e^2$ , the error function-type Coulomb force

$$V_D(r) = \frac{\alpha}{r} \text{erf}(\beta r) \quad (6.3)$$

in Eq. (6.1) is the direct potential of the  $\alpha$ - $\alpha$  RGM. When a simple  $(0s)^4$  harmonic-oscillator shell-model wave function with the width parameter  $\nu$  is assumed for the  $\alpha$ -cluster, the parameter  $\beta$  is expressed as

$$\beta = \sqrt{\frac{\nu}{\left(1 - \frac{1}{2\mu}\right)}} = 2\sqrt{\nu/3} , \quad (6.4)$$

where  $\mu = 4 \cdot 4 / (4 + 4) = 2$  is the reduced mass number of the  $\alpha\alpha$  system. On the other hand, the rms radius of the  $\alpha$ -cluster with  $A = 4$  is given by

$$r_\alpha = \sqrt{\langle r^2 \rangle_\alpha} = \sqrt{\frac{3}{4} \left(1 - \frac{1}{A}\right) \frac{1}{\nu}} = \frac{3}{4} \frac{1}{\sqrt{\nu}}, \quad (6.5)$$

without the proton size effect, so that  $\beta$  is related to  $r_\alpha$  through

$$\beta = \frac{\sqrt{3}}{2 \cdot r_\alpha}. \quad (6.6)$$

In ABd,  $r_\alpha = 1.44$  fm is assumed, corresponding to  $\nu = 0.271$  fm<sup>-2</sup>.

In the momentum representation, we use the sharply cut-off Coulomb force at the nucleon level. The corresponding direct  $\alpha\alpha$  potential is given by

$$V_D^\rho(r) = \frac{\alpha}{r} \left\{ \operatorname{erf}(\beta r) - \frac{1}{2} [\operatorname{erf}(\beta(r + \rho)) + \operatorname{erf}(\beta(r - \rho))] \right\}. \quad (6.7)$$

If we use this screened Coulomb potential in Eq. (6.1), we find

$$V_{\alpha\alpha}^\rho(r) = V_1 e^{-\eta_1 r^2} + V_2 e^{-\eta_2 r^2} + V_D^\rho(r). \quad (6.8)$$

Here, we separate  $V_D^\rho(r)$  into

$$\begin{aligned} V_D^\rho(r) &= \frac{\alpha}{r} \left\{ [\operatorname{erf}(\beta r) - 1] + 1 - \frac{1}{2} [\operatorname{erf}(\beta(r + \rho)) + \operatorname{erf}(\beta(r - \rho))] \right\} \\ &= -\frac{\alpha}{r} [1 - \operatorname{erf}(\beta r)] + \frac{\alpha}{r} \alpha_\rho(r), \end{aligned} \quad (6.9)$$

and set

$$\alpha_\rho(r) = 1 - \frac{1}{2} [\operatorname{erf}(\beta(r + \rho)) + \operatorname{erf}(\beta(r - \rho))]. \quad (6.10)$$

Then, the  $\alpha\alpha$  potential which should be used in the momentum representation becomes

$$\begin{aligned} V_{\alpha\alpha}^\rho(r) &= V(r) + \frac{\alpha}{r} \alpha_\rho(r) \\ \text{with} \quad V(r) &= V_1 e^{-\eta_1 r^2} + V_2 e^{-\eta_2 r^2} + \mathcal{W}(r). \end{aligned} \quad (6.11)$$

Here,  $\mathcal{W}(r) = -(\alpha/r) [1 - \operatorname{erf}(\beta r)]$  is the short-range attraction originating from the Coulomb potential. In fact, the asymptotic expansion of the error function yields

$$\mathcal{W}(r) = \frac{\alpha}{r} [\operatorname{erf}(\beta r) - 1] \sim -\frac{\alpha}{r} e^{-(\beta r)^2} \sum_{n=0}^{\infty} (-)^n \frac{(2n-1)!!}{2^{n+1}} \left(\frac{1}{\beta r}\right)^{2n+1}. \quad (6.12)$$

We find that  $\mathcal{W}(r)$  is sufficiently small around  $(\beta r)^2 \sim 16$ ; namely,  $r \sim 4/\beta \sim 7$  fm. (Actually, even around  $\sim 4$  fm, as seen in Fig. 2 below.)

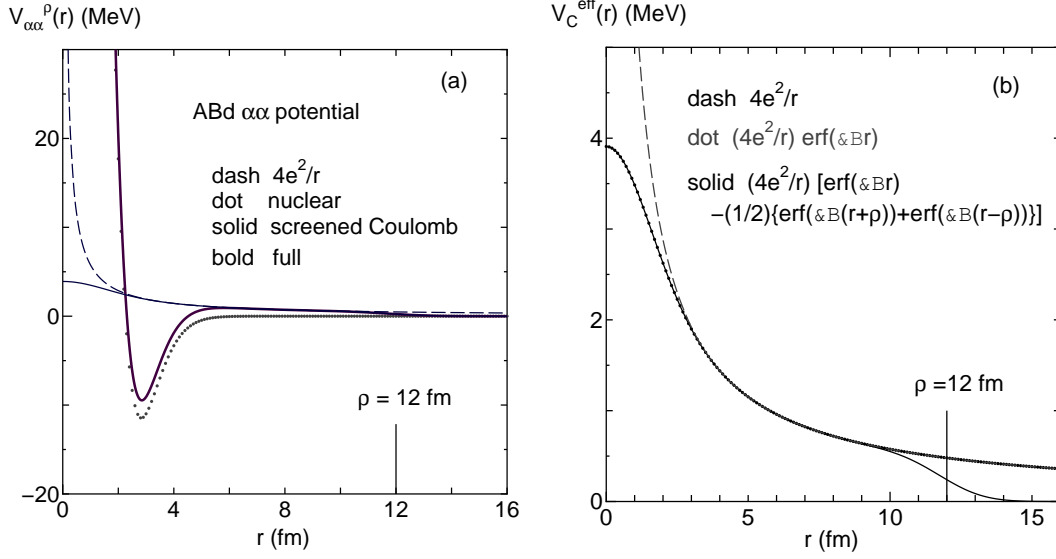


Fig. 1. (a):  $S$ -wave Ali-Bodmer potential ABd with the screened Coulomb force (solid curve) for the  $\alpha\alpha$  system. The dashed curve denotes the simple Coulomb potential  $V_C(r) = 4e^2/r$  and the dotted curve the nuclear part. The full potential is shown in the bold solid curve. The cut-off radius of the sharply cut-off Coulomb force at the nucleon level is assumed to be  $\rho = 12$  fm. (b): The enlarged profiles of (a) for the various Coulomb potentials. The solid curve denotes the screened Coulomb direct potential Eq. (6.7) with the error function form.

We illustrate in Fig. 1(a) the  $S$ -wave Ali-Bodmer potential ABd and in (b) the enlarged profiles of various types of Coulomb potentials. The cut-off function  $\alpha_\rho(r)$  in Eq. (6.10) for the cut-off Coulomb radius  $\rho = 12$  fm and the short-range Coulomb potential  $\mathcal{W}(r)$  in Eq. (6.12) are shown in Fig. 2. We find that  $\alpha_\rho(r)$  satisfies the conditions 1) - 3) of the screened Coulomb potential. In particular, the much stringent condition 3)' in Eq. (4.5) is also satisfied with the smoothness parameter  $b \sim 3$  fm. If we take  $b = 6$  fm, the deviation of  $\alpha_\rho(r)$  from 1 (or 0) at  $R_{\text{in}} = \rho - b = 6$  fm (or at  $R_{\text{out}} = \rho + b = 18$  fm) is less than  $10^{-6}$ . Note that this kind of a rapid transition from 1 to 0 is not achieved in the standard screening functions in the form of  $\alpha_\rho(r) = e^{-(r/\rho)^n}$ , unless  $n$  is taken to be very large like  $n \geq 20$ . In this sense, our screened Coulomb potential is a small deviation from the sharply cut-off Coulomb potential, which is probably related to the smallness of the limit  $\lim_{\rho \rightarrow \infty} A_\ell^\rho \sim 0$  if it exists. This property must also be related to the small deviation of the shift function  $\zeta^\rho(k)$  in Eq. (4.13) from  $\eta \log(2k\rho)$ , which is the result of the sharply cut-off Coulomb potential in Eq. (2.27). We will show in Appendix B that the screening function  $\alpha_\rho(r)$  in Eq. (6.10) satisfies the limit

$$\zeta^\rho(k) \rightarrow \eta \log(2k\rho) \quad \text{as } \rho \rightarrow \infty, \quad (6.13)$$

in contrast to the  $\alpha_\rho(r) = e^{-(r/\rho)^n}$  case. In the latter case, the right-hand side of Eq. (6.13) contains an extra constant term  $-(\eta/n)\gamma$  with  $\gamma$  being the Euler constant. (See Eq. (B.3).)

First, we have neglected the nuclear potential  $V_1 = 0$  and  $V_2 = 0$  in Eq. (6.1)

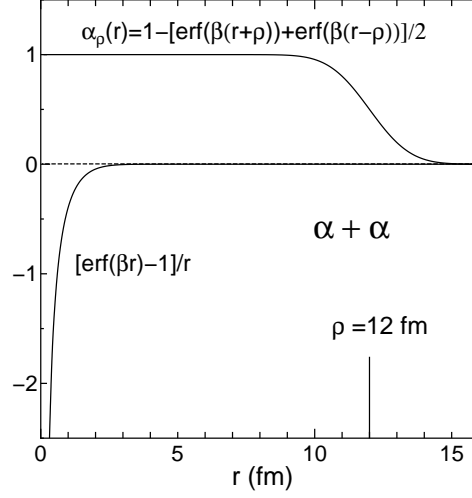


Fig. 2. The cut-off function  $\alpha_\rho(r)$  in Eq. (6.10) with the cut-off Coulomb radius  $\rho = 12$  fm and the short-range Coulomb potential  $\mathcal{W}(r)/\alpha = [\text{erf}(\beta r) - 1]/r$  in Eq. (6.12) for the  $\alpha\alpha$  screened Coulomb potential (in the unit of  $\text{fm}^{-1}$ ).

and compared the nuclear phase shifts between the present method and the direct method using Eq. (1.1). In the direct method, the relative wave function  $\psi_\ell(r)$  in Eq. (1.1) is solved from  $r = 0$  to  $R_{\text{out}} = 12 + 6 = 18$  fm by the RKG method and smoothly connected to a linear combination of the pure Coulomb wave functions at  $r = R_{\text{out}}$ . Since we are using the error function Coulomb, the nuclear phase shift does not become zero. In the  $S$ -wave,  $\delta_0^N$  increases from 0 to  $11.088^\circ$ , when the energy increases up to  $E_{\text{cm}} = 15$  MeV. Similarly,  $\delta_2^N = 0.473^\circ$  and  $\delta_4^N = 0.013^\circ$  at  $E_{\text{cm}} = 15$  MeV. In the momentum-space approach, we first solve the LS equation and calculate  $\delta_\ell^\rho$  (which is the screened Coulomb phase shift) by assuming  $\rho = 12$  fm. The phase shift is then transformed to  $\delta_\ell^N$  through the connection condition Eq. (4.24). Here, we assumed  $b = 6$  fm, and  $\tilde{F}_\ell^\rho(k, r)$ ,  $\tilde{G}_\ell^\rho(k, r)$  are calculated from  $R_{\text{in}} = 12 - 6 = 6$  fm to  $R_{\text{out}} = 12 + 6 = 18$  fm, also by the RKG method, with the pure Coulomb values at  $R_{\text{in}} = 6$  fm as the starting values. The results by these two different methods, of course, agree to each other completely within the numerical accuracy less than  $0.001^\circ$ . Next, we switch on  $V_1$  and  $V_2$  and repeated the same calculations. The result is shown in Table I. For each incident energy, the first row indicates solutions obtained by the RKG method, and the second row those in the momentum-space approach. Only different figures from the first row are shown. In the left-hand side, the final results of  $\delta_\ell^N$  are compared. In the right-hand side, the phase shifts  $\bar{\delta}_\ell^\rho$  directly obtained from the LS equation (before the transformation) are also compared. We find that, in the lowest energy  $E_{\text{cm}} = 1$  MeV, a difference of  $0.005^\circ$  exists both in  $\delta_\ell^N$  and  $\bar{\delta}_\ell^\rho$ . This is probably the inaccuracy of solving the LS equation in the low energies. For other energies, the difference is less than  $0.001^\circ$ , and the agreement of the results obtained by our method with the exact solutions is quite satisfactory.

Table I. Comparison of the  $\alpha\alpha$  nuclear phase shifts ( $\delta_\ell^N$ ) of the ABd potential with the direct method. For each cm energy  $E_{\text{cm}}$ , the first row indicates solutions by the RKG method, connected at  $R_{\text{out}} = 18$  fm by Eq. (1.1). The second row stands for the solutions by the present momentum-space approach. Only different figures from the first row are shown. In the left-hand side, the final results of  $\delta_\ell^N$  are compared. In the right-hand side, the phase shifts  $\bar{\delta}_\ell^\rho$  directly obtained from the LS equation (before the transformation) are also compared. The cut-off Coulomb radius  $\rho$  is chosen to be  $\rho = 12$  fm and the smoothness parameter in Eq. (4.5) is  $b = 6$  fm.  $(\hbar^2/M_N) = 41.786$  MeV  $\cdot$  fm<sup>2</sup> and  $e^2 = 1.44$  MeV  $\cdot$  fm are used.

$E_{\text{cm}}$ (MeV)	$\delta_\ell^N$			$\bar{\delta}_\ell^\rho$		
	$S$	$D$	$G$	$S$	$D$	$G$
1	147.021	0.485	0.000	40.403	152.767	178.017
	16			398		
2	110.751	9.736	0.008	10.133	141.754	168.315
	2	5		4		
3	85.082	66.577	0.065	176.887	22.894	158.138
		6			3	
4	65.251	109.426	0.261	165.105	65.200	153.102
5	49.031	115.142	0.748	153.868	73.736	153.013
6	35.271	113.611	1.756	144.160	73.663	154.882
7	23.299	110.142	3.661	135.833	71.241	157.196
8	12.689	106.077	7.140	128.379	68.336	160.738
9	3.153	101.875	13.575	121.482	65.314	167.386
10	-5.513	97.719	26.095	115.039	62.204	0.375
11	-13.462	93.685	50.814	109.027	59.060	25.569
12	-20.807	89.805	86.674	103.419	55.958	61.717
13	-27.637	86.086	113.376	98.168	52.954	88.789
						8
14	-34.024	82.526	127.388	93.223	50.067	103.178
15	-40.025	79.119	134.966	88.538	47.293	111.085
					2	

## 6.2. $\alpha\alpha$ Lippmann-Schwinger RGM by the Minnesota three-range force

As a more complex system, we apply the present method to the  $\alpha\alpha$  LS-RGM, using the Minnesota three-range force. In this calculation, we solve the RGM equation in the momentum space. All the Born kernels including the direct term and the RGM exchange kernels for the sharply cut-off Coulomb force between two protons

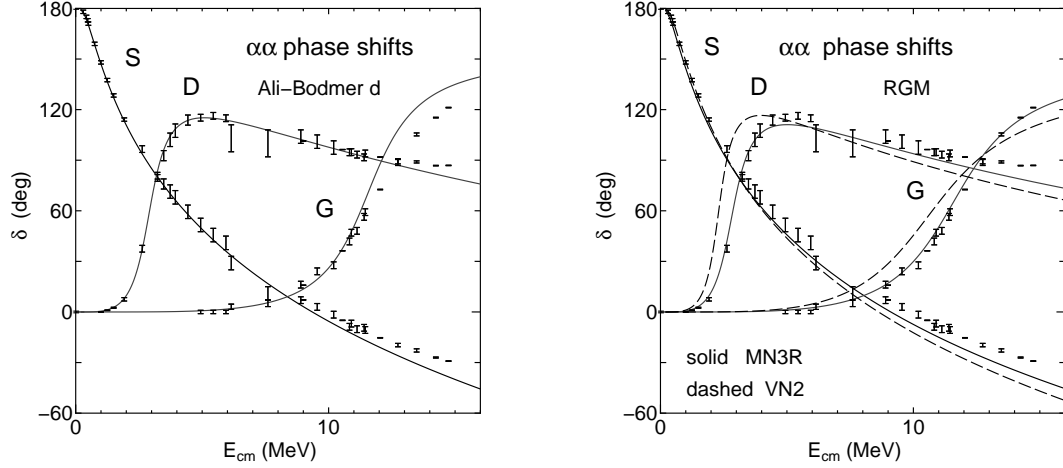


Fig. 3. (a):  $S$ -,  $D$ - and  $G$ -wave  $\alpha\alpha$  phase shifts predicted by the Ali-Bodmer potential ABd. (b): The same as (a), but for  $\alpha\alpha$  RGM by Volkov No. 2 (VN2) with  $m = 0.59$  and  $\nu = 0.275 \text{ fm}^{-1}$  (dashed curves) and by Minnesota three-range (MN3R) potentials with  $u = 0.94687$  and  $\nu = 0.257 \text{ fm}^{-1}$  (solid curves), the latter result is in better agreement with experiment.

are analytically calculated. For example, the direct Born kernels of the error function Coulomb potential in Eq. (6.3) and the screened Coulomb potential in Eq. (6.7) are given by

$$M_D^{CL}(\mathbf{q}_f, \mathbf{q}_i) = \langle e^{i\mathbf{q}_f \cdot \mathbf{r}} | \frac{4e^2}{r} \text{erf}(\beta r) | e^{i\mathbf{q}_i \cdot \mathbf{r}} \rangle = 4e^2 \frac{4\pi}{k^2} e^{-\frac{1}{4} \left( \frac{k}{\beta} \right)^2},$$

$$M_D^{\rho CL}(\mathbf{q}_f, \mathbf{q}_i) = \langle e^{i\mathbf{q}_f \cdot \mathbf{r}} | V_D^\rho(r) | e^{i\mathbf{q}_i \cdot \mathbf{r}} \rangle = 4e^2 2\pi \rho^2 \left( \frac{\sin \frac{k\rho}{2}}{\frac{k\rho}{2}} \right)^2 e^{-\frac{1}{4} \left( \frac{k}{\beta} \right)^2}, \quad (6.14)$$

where  $\mathbf{k} = \mathbf{q}_f - \mathbf{q}_i$ . Note that  $M_D^{CL}(\mathbf{q}_f, \mathbf{q}_i)$  involves the Coulomb singularity at  $|\mathbf{q}_f| = |\mathbf{q}_i|$ , while  $M_D^{\rho CL}(\mathbf{q}_f, \mathbf{q}_i)$  does not have such a singularity. A numerical challenge is the angular momentum projection of this kernel. We have used a standard Gauss-Legendre integration quadrature, taking many discretization points. We can check the accuracy of this numerical integration by examining the redundancy condition of the Pauli forbidden states for the  $S$ - and  $D$ -waves. Various cut-off Coulomb parameters are chosen from  $\rho = 8 \text{ fm}$  to  $16 \text{ fm}$ , with  $b = 6 \text{ fm}$  fixed. The modified Coulomb wave functions are therefore solved from  $R_{in} = \rho - 6 \text{ fm}$  to  $R_{out} = \rho + 6 \text{ fm}$ . In Table 2, we list the variation of the nuclear phase shifts, depending on the choice of  $\rho$ . We find that the results are quite stable in this appropriate range of  $\rho$ . We show in Fig. 3(a) the  $\alpha\alpha$  phase shifts predicted by Ali-Bodmer d potential and in Fig. 3(b) the results by the LS-RGM using the Minnesota three-range force and the Volkov No. 2 two-range force.

### 6.3. $pd$ elastic scattering

As in the case of the  $\alpha\alpha$  scattering discussed in the preceding subsections, the screening function  $\alpha^\rho(R)$  for the  $pd$  elastic scattering should be derived in a consistent



Table II. Cut-off radius ( $\rho$ ) dependence of the nuclear phase shifts  $\delta_\ell$  for the  $\alpha\alpha$  Lippmann-Schwinger RGM. The Minnesota three-range force with  $u = 0.94687$  and  $\nu = 0.257 \text{ fm}^{-2}$  are used.

$E_{\text{cm}}$ (MeV)	$\rho$ (fm)				
	8	10	12	14	16
$^1S_0$	1.000	144.448	144.450	144.450	144.450
	2.000	107.561	107.563	107.563	107.563
	3.000	81.689	81.690	81.690	81.690
	4.000	61.847	61.848	61.848	61.848
	5.000	45.740	45.741	45.741	45.741
	6.000	32.176	32.177	32.177	32.177
	8.000	10.145	10.146	10.146	10.146
	0.933				
	10.000	-7.395	-7.394	-7.394	-7.394
	12.000	-21.988	-21.987	-21.987	-21.987
	15.000	-40.163	-40.163	-40.163	-40.163
$^1D_2$	1.000	0.589	0.590	0.590	0.587
	2.000	11.641	11.644	11.644	11.645
	3.000	70.120	70.134	70.134	70.134
	4.000	106.361	106.365	106.365	106.365
	5.000	111.080	111.081	111.081	111.081
	6.000	109.344	109.345	109.345	109.345
	8.000	101.886	101.887	101.887	101.887
	10.000	93.853	93.854	93.854	93.854
	12.000	86.367	86.368	86.368	86.368
	15.000	76.354	76.355	76.355	76.355
$^1G_4$	1.000	0.000	0.000	0.000	0.000
	2.000	0.013	0.013	0.013	0.014
	3.000	0.102	0.102	0.102	0.102
	4.000	0.398	0.399	0.399	0.399
	5.000	1.107	1.108	1.108	1.108
	6.000	2.522	2.523	2.523	2.523
	8.000	9.514	9.517	9.517	9.517
	10.000	29.924	29.928	29.928	29.928
	12.000	75.818	75.820	75.820	75.820
	15.000	120.744	120.745	120.745	120.745

way with the screened Coulomb potential between two protons in Eq. (5.1). In our application of the quark-model baryon-baryon interaction fss2 to the  $pd$  elastic scattering in Ref. 29), the sharply cut-off Coulomb force is introduced at the quark level in the form of  $(1/r_{qq})\theta(\rho - r_{qq})$ , where  $r_{qq}$  is the relative distance between two quarks. The proton-proton ( $pp$ ) potential  $\omega^\rho(r)$  is obtained by folding it with the

(3q)-(3q) internal wave function, resulting in

$$\omega^\rho(r; 1, 2) = \frac{e^2}{r} \left\{ \operatorname{erf} \left( \frac{\sqrt{3} r}{2 b} \right) - \frac{1}{2} \left[ \operatorname{erf} \left( \frac{\sqrt{3} r + \rho}{2 b} \right) + \operatorname{erf} \left( \frac{\sqrt{3} r - \rho}{2 b} \right) \right] \right\} \\ \times \frac{1 + \tau_z(1)}{2} \frac{1 + \tau_z(2)}{2}. \quad (6.15)$$

where  $r$  is the distance between the two protons,  $r = |\mathbf{r}_{12}| = |\mathbf{x}_1 - \mathbf{x}_2|$  and  $b$  is the harmonic-oscillator range parameter of the (3q)-clusters. Note that this screened Coulomb potential for the two protons is not equal to Eq. (5.1) with a mere change of  $\alpha^\rho(r)$  to Eq. (6.10) (with a trivial modification  $\beta \rightarrow (\sqrt{3}/2b)$ ), but also contains the contributions from the short-range Coulomb potential in Eq. (6.12). We calculate the  $pd$  screened Coulomb potential by further folding the  $pp$  potential in Eq. (6.15) with the deuteron wave function  $\langle \mathbf{r}; 1, 2 | \psi_d \rangle$ :

$$V_{pd}^{\rho C}(R) = \langle \psi^d | \omega^\rho(|\mathbf{R} + \mathbf{r}/2|; 2, 3) | \psi^d \rangle + \langle \psi^d | \omega^\rho(|\mathbf{R} - \mathbf{r}/2|; 3, 1) | \psi^d \rangle, \quad (6.16)$$

where  $\mathbf{R} = \mathbf{x}_3 - (\mathbf{x}_1 + \mathbf{x}_2)/2$  is the relative coordinate between the center-of-mass of the deuteron and the proton. This calculation is made in Appendix C. We assign the long-range part of  $V_{pd}^{\rho C}(R)$  in Eq. (C.2) to  $W^\rho(R)$  in Eq. (5.2), and parametrize it as  $W^\rho(R) = (e^2/R)\alpha^\rho(R)$ . The screening function  $\alpha^\rho(R)$  is numerically calculated by using Eqs. (C.15) - (C.17) and the momentum-space deuteron wave function expanded in the dipole form factors.<sup>27)</sup> Here, we only show in Fig. 4 the profiles of the screening function  $\alpha^\rho(R)$  and the short-range Coulomb potential (the polarization potential)  $\mathcal{W}(R)$  for the simplest deuteron channel with  $J^\pi = 1/2^+$ . We find that the cut-off behavior around  $R \sim \rho$  is fairly sharp even in  $\rho \sim 8$  fm. The short-range Coulomb potential  $\mathcal{W}(R)$  is  $\rho$ -independent as shown in Eq. (C.5). The coupling potential  $W^\rho(R)$  between different channel-spin states,  $(\ell S_c) \neq (\ell' S'_c)$ , is very small. We therefore neglect this and solve the screened Coulomb problem only by using the diagonal part of  $(\ell S_c)$ , in order to generate the regular and irregular screened Coulomb wave functions for the connection condition.

Some typical eigenphase shifts of the  $E_p = 65$  MeV  $pd$  scattering with the Coulomb cut-off radius  $\rho = 8, 16$  and  $20$  fm are listed in Table III for the  $S$  and  $P$  waves. Here, we have assumed the maximum total angular-momentum of the two-nucleon subsystem,  $I_{\max} = 4$ . The real parts of the eigenphase shifts are only given for simplicity. We find that the inclusion of the cut-off Coulomb force gives an apparent repulsive effect, namely, the  $S$ -wave and  $P$ -wave eigenphase shifts are  $-0.9^\circ \sim -2.5^\circ$  ( $-2.5^\circ \sim -3.1^\circ$ ) more repulsive than in the no Coulomb case, if  $\rho = 8$  fm ( $\rho = 16$  fm) is assumed. The transformation by the connection condition in Eq. (5.28) gives an attractive effect to make the resultant eigenphase shifts rather close to the no Coulomb case. As long as the low partial waves such as the  $S$  and  $P$  waves are concerned, the final results of the nuclear eigenphase shifts are rather stable within the fluctuation of less than  $0.8^\circ$ . We have calculated  $pd$  differential cross sections and other polarization observables, using various  $\rho$  values. The results by  $\rho = 8$  fm is quite reasonable, but if we take larger values like  $\rho = 16$  and  $20$  fm, we have found that unpleasant oscillations develop in all the observables.

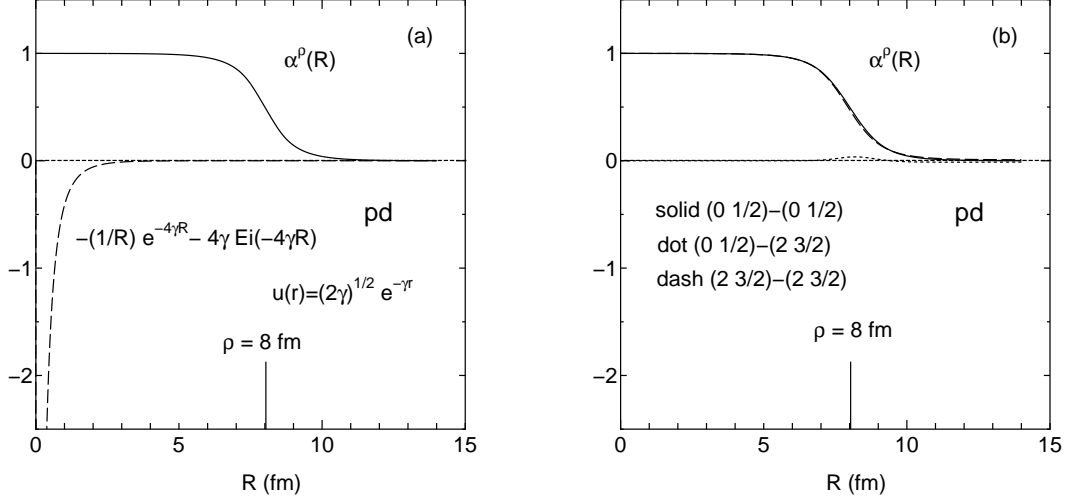


Fig. 4. (a) The screening function  $\alpha^\rho(R)$  (solid curve) and the short-range potential  $W(R)/e^2$  (dashed curve) (in the unit of  $\text{fm}^{-1}$ ), given in Eq. (C-7) for the  $pd$  scattering. A simple  $S$ -wave deuteron wave function  $u(r) = \sqrt{2\gamma} e^{-\gamma r}$  is used. The Coulomb cut-off radius is  $\rho = 8$  fm. (b) The realistic  $\alpha^\rho(R)$  for the  $pd$  scattering in the simplest deuteron channel with  $J^\pi = 1/2^+$ . In the  $(\ell S_c) = (0\ 1/2)$  (solid curve) and  $(2\ 3/2)$  (dashed curve) diagonal channels, curves are almost identical. The off-diagonal  $\alpha^\rho(R)$  with  $(\ell S_c) - (\ell' S'_c) = (0\ 1/2) - (2\ 3/2)$  is very small. Here,  $\ell$  is the relative angular-momentum between  $p$  and  $d$  and  $S_c$  is the channel spin.

The origin of the oscillations is traced back to the high partial waves, in which the restriction of  $I_{\max} = 4$  is too severe. Since we are using the channel-spin formalism,

Table III. Real parts of the nuclear eigenphase shifts for the  $Nd$  elastic scattering at  $E_N = 65$  MeV. The  $nd$  phase shifts with no Coulomb force and the  $pd$  phase shifts including the cut-off Coulomb force with  $\rho = 8, 16$  and  $20$  fm are listed. For  $\rho = 8$  fm (before), the eigenphase shifts before transformation in Eq. (5-28) are also shown. The maximum total angular-momentum of the two-nucleon subsystem is  $I_{\max} = 4$ , and the momentum discretization points  $n = n_1 - n_2 - n_3 = 6-6-5$  are used in the definition shown in Ref. 28).

$^{2S+1}\ell_J$	no Coulomb	with Coulomb			
		$\rho = 8$ fm (before)	$\rho = 8$ fm	$\rho = 16$ fm	$\rho = 20$ fm
$^2S_{1/2}$	26.84	24.38	28.70	28.99	29.01
$^4D_{1/2}$	-7.25	-9.41	-7.20	-6.92	-6.76
$^2P_{1/2}$	-0.44	-2.51	-0.04	0.35	0.45
$^4P_{1/2}$	24.28	21.83	24.76	24.98	24.99
$^4S_{3/2}$	32.11	31.23	33.79	34.25	34.63
$^2D_{3/2}$	8.74	6.87	9.15	9.35	9.51
$^4D_{3/2}$	-5.49	-7.84	-5.32	-5.01	-5.11
$^4P_{3/2}$	24.98	22.53	25.32	25.84	25.72
$^2P_{3/2}$	6.73	4.60	7.16	7.38	7.47
$^4F_{3/2}$	-1.05	-2.81	-0.86	-0.63	-0.55

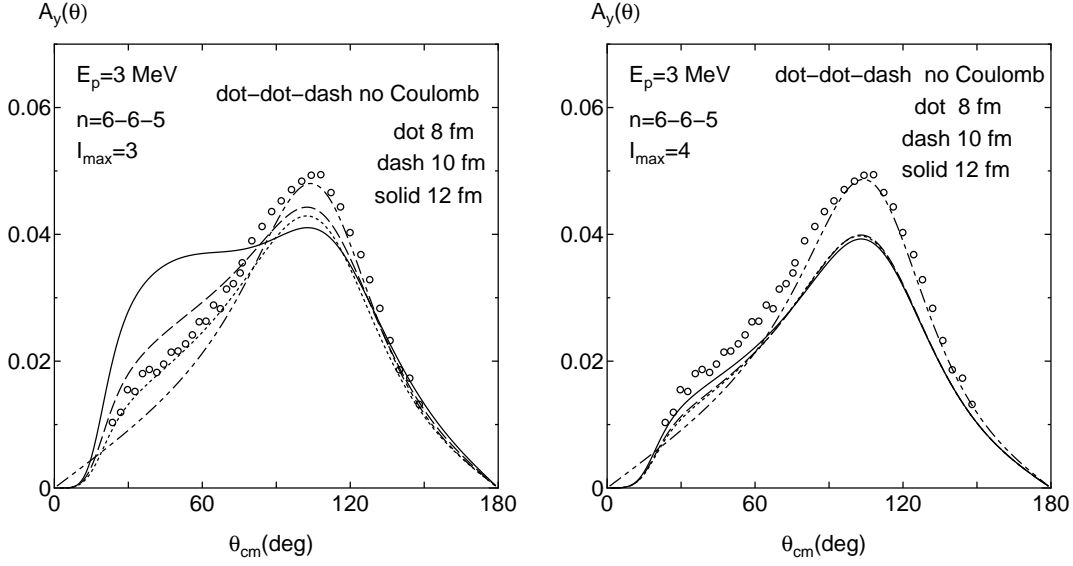


Fig. 5. Cut-off radius dependence of the proton analyzing power for the  $pd$  elastic scattering at  $E_p = 3$  MeV. The results of no Coulomb case,  $\rho = 8, 10$ , and  $12$  fm are shown by the dot-dot-dashed, dotted, dashed, and solid curves, respectively. The left panel shows the results of  $I_{\max} = 3$  and the right panel of  $I_{\max} = 4$ . The  $pd$  experimental data from Ref. 32) are also shown by circles.

the total angular momentum  $J^\pi$  of the three nucleon system is achieved by the angular-momentum coupling  $(\ell S_c)J$ , where the channel spin  $S_c$  is constructed from  $(I_{\frac{1}{2}})S_c$ . For a large  $J^\pi$ , a large contribution of the Coulomb force from the large relative orbital angular momentum of the two-proton subsystem is not fully taken into account, since the magnitude of  $S_c$  is restricted by  $I_{\max} = 4$ . To demonstrate this situation, we show in Fig. 5 the  $\rho$ -dependence of the nucleon analyzing power for the 3 MeV  $pd$  scattering, calculated with  $I_{\max} = 3$  and  $I_{\max} = 4$ . In the forward angular region with  $\theta_{cm} < 90^\circ$ , we find that unpleasant bump structure develops as  $\rho$  increases from 8 fm to 12 fm, when  $I_{\max} = 3$  is used. However, such enhancement is strongly suppressed when  $I_{\max} = 4$  is used. This demonstrates very clearly that two-nucleon partial waves should be included up to sufficiently higher values to obtain the well converged results, if the screened Coulomb force is incorporated into the standard AGS equations.

Since the calculation with  $I_{\max} = 6$  and more is not presently possible because of the computer resources, here we propose to cut the Coulomb force for higher  $J^\pi$  values and use a simple ‘‘Coulomb externally corrected approximation’’, in which the  $nd$  eigenphase shifts are directly used for the nuclear phase shifts.<sup>31)</sup> Figure 6 shows the  $pd$  differential cross sections and some polarization observables at  $E_p = 65$  MeV, calculated by neglecting the Coulomb force for  $J^\pi \geq 11/2^+$ . We find that the results with  $\rho = 8, 16$  and  $20$  fm are very similar, although some difference is seen in  $T_{22}(\theta)$ . The results with  $\rho = 8$  fm are almost the same as the full calculation including the Coulomb force to all the partial waves.

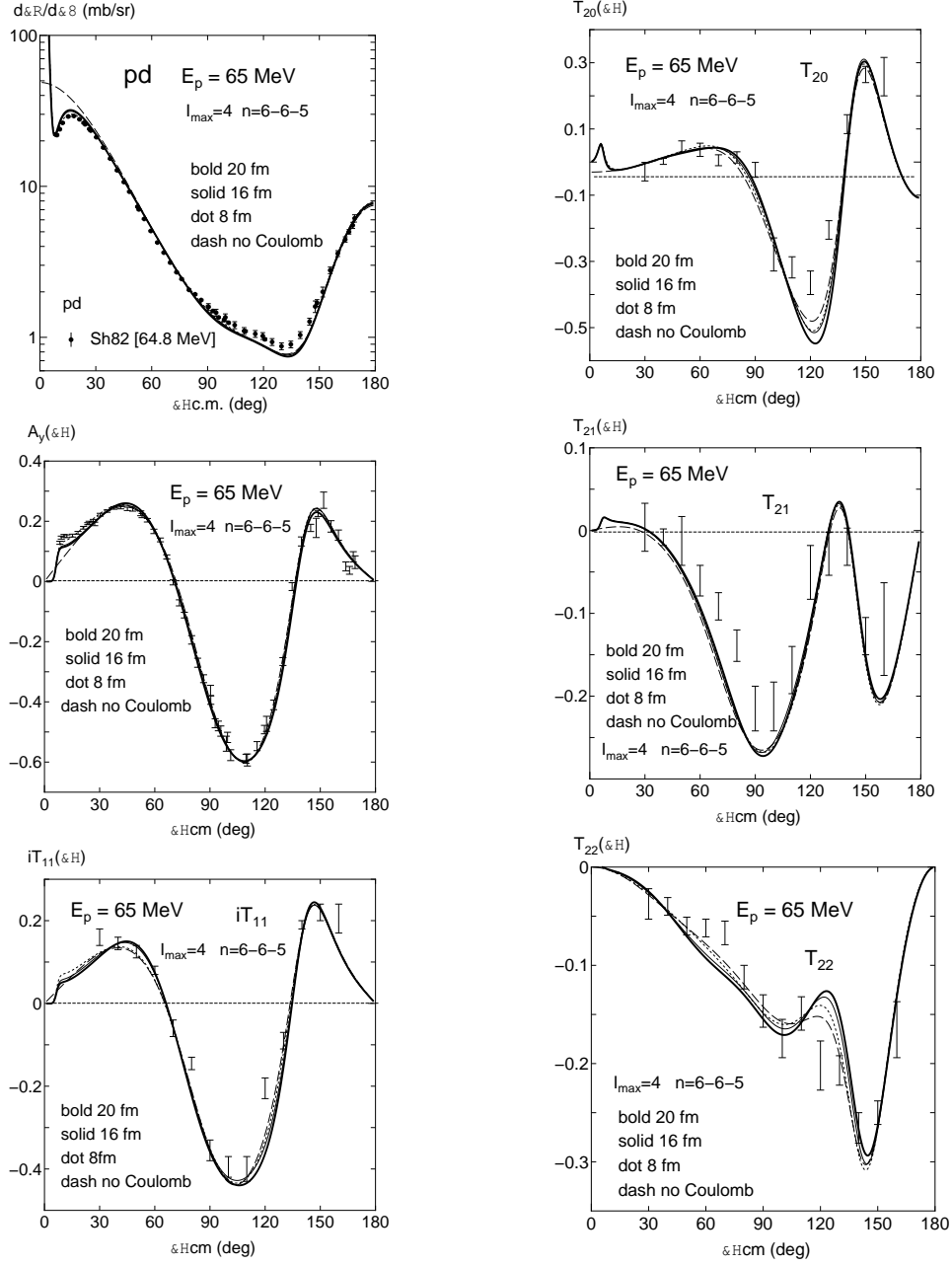


Fig. 6.  $pd$  differential cross sections ( $d\sigma/d\Omega$ ), analyzing power ( $A_y(\theta)$ ) of the proton, and vector-type ( $iT_{11}(\theta)$ ) and tensor-type ( $T_{2m}(\theta)$ ) analyzing powers of the deuteron at  $E_p = 65$  MeV. The results with no Coulomb case,  $\rho = 8, 16$ , and  $20$  fm are shown by the dashed, dotted, solid, and bold-solid curves, respectively. These curves almost overlap with each other, except for the forward nuclear-Coulomb interference region. The screened Coulomb force is neglected for higher partial waves with  $J^\pi \geq 11/2^+$ . The experimental data are taken from Ref. 33) for  $d\sigma/d\Omega$  and  $A_y(\theta)$ , and from Ref. 34) for  $iT_{11}(\theta)$  and  $T_{2m}(\theta)$ .

## §7. Summary and outlook

In the present work we have proposed a practical method to deal with the Coulomb problem in the momentum space. Although the standard procedure to deal with the Coulomb force in two-body systems is formulated in the configuration space, the extension of such an approach to three-body systems is not trivial.<sup>30)</sup> Here, we have reformulated the momentum-space approach of the two-cluster systems based on the essential idea of the “screening and renormalization procedure”, which is recently used in the standard formulation of the AGS equations for the  $pd$  scattering in the momentum representation.<sup>7),8),9)</sup> In this approach, the screened Coulomb force with a cut-off parameter  $\rho$  is introduced to the basic equations as if it is a part of the short-range nuclear force. The two-potential formula for the short-range potentials is used to generate the scattering amplitude. The pure Coulomb results are reproduced by taking  $\rho \rightarrow \infty$  limit, based on the Taylor’s formula<sup>15),16)</sup> for the phase renormalization of the asymptotic wave functions of the screened Coulomb potential. The central issue in this approach is if one can reproduce the exact Coulomb results by taking a finite  $\rho$ . Since the quasi-singular nature of the screened Coulomb force becomes stronger for larger  $\rho$ , it is essential that one can reproduce the almost *exact* results with a reasonable choice of  $\rho$ .

To achieve this, we propose to extend the Vincent and Phatak approach,<sup>6)</sup> which is originally formulated for the sharply cut-off Coulomb problems. When a sharply cut-off Coulomb force with the cut-off radius  $\rho$  is introduced at the level of constituent particles, two-cluster direct potential of the Coulomb force becomes in general a local screened Coulomb potential implemented with the short-range Coulomb force. The screening function  $\alpha^\rho(r)$  is determined by the properties of the cluster wave functions, and involves a smoothness parameter  $b$  related to the size of clusters. In practice,  $b$  satisfies  $b \ll \rho$ , which is an additional condition to the Taylor’s properties<sup>15)</sup> of screening functions. We find that this condition is necessary to make the present treatment work well. We pay attention to the existence of two different types of asymptotic waves contained in the screened Coulomb wave functions. The first one is the approximate Coulomb wave for the relative distance of two clusters,  $r$ , smaller than  $R_{\text{in}} = \rho - b$ , and the other is the free (no-Coulomb) wave in the longer range region,  $r > R_{\text{out}} = \rho + b$ . The asymptotic Hamiltonian composed of the screened Coulomb force allows us to calculate constant Wronskians of this Hamiltonian in either region. Using this property, we can extend the standard procedure of matching conditions for asymptotic waves to the screened Coulomb potential.

We should note that the renormalization property of the screened Coulomb wave functions is more involved than in the sharply cut-off Coulomb case. In particular, the irregular function of the screened Coulomb potential in general contains an admixture of the regular solution even in the  $\rho \rightarrow \infty$  limit. As the result, the limit of the Green function for the screened Coulomb potential is not reduced to the Coulomb Green function. This requires an extra renormalization of the regular wave function for the problem of the short-range nuclear potential plus the screened Coulomb potential. This renormalization factor, however, does not affect the final expression of the connection condition, since it is given by the ratio of Wronskians.

We have applied this method first to an exactly solvable model of the  $\alpha\alpha$  scattering with the Ali-Bodmer potential and confirmed that essentially *exact* phase shifts are reproduced, using a finite  $\rho$ . The stability of nuclear phase shifts with respect to the change of  $\rho$  in some appropriate range is demonstrated by using the  $\alpha\alpha$  Lippmann-Schwinger RGM with the Minnesota three-range force. In the application to the  $pd$  elastic scattering, some dependence on the choice of  $\rho$  remains although the essential features of the nuclear and Coulomb interference in forward angles are reproduced not only for the differential cross sections but also for the deuteron tensor analyzing powers.

We have to admit that the completely satisfactory Coulomb treatment of the three-body system is still beyond the way. First, the stability of  $\rho$  in the case of the above  $pd$  elastic scattering is not completely realized. We have examined all the observables for the  $pd$  elastic scattering in the energy range  $E_p \leq 65$  MeV, and found that the present choice  $\rho \sim 8 - 9$  fm is a reasonable choice to reproduce almost all the experimental data.<sup>29)</sup> The forward behavior of the vector analyzing power  $A_y(\theta)$  for the proton and  $iT_{11}(\theta)$  for the deuteron is not consistently achieved in the low-energy region, using a unique  $\rho$ . Choosing much larger  $\rho$  around  $\rho \sim 16 - 20$  fm is almost prohibited since the solution of the AGS equation becomes very singular and the partial waves included in the actual calculations are restricted by the hardware. Another problem is the treatment of the Coulomb force in the breakup processes.<sup>35)</sup> The phase renormalization for the observed two protons at the final stage is not trivial because of the exchange breakup amplitude. We probably need to solve the Coulomb-modified AGS equations in spite of the very singular nature of the screened Coulomb wave functions in the momentum representation. Finally, we mention that the Coulomb treatment of three charged particles like the three  $\alpha$  system is a big challenge, since the asymptotic behavior of the three charged particles is not *a priori* known.

### Acknowledgements

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### Appendix A

#### —— Definition of the Coulomb wave functions ——

The usual regular solution  $\psi_\ell(k, r)$  and the regular solution corresponding to the Jost solution,  $\varphi_\ell(k, r)$ , for the Coulomb problem are defined by the confluent

hypergeometrical functions through

$$\begin{aligned}
\varphi_\ell(k, r) &= \frac{1}{k^\ell} F_\ell(k) \psi_\ell(k, r) \\
&= \frac{r^{\ell+1}}{(2\ell+1)!!} e^{ikr} F(\ell+1+i\eta, 2\ell+2, -2ikr) \\
&= \frac{r^{\ell+1}}{(2\ell+1)!!} e^{-ikr} F(\ell+1-i\eta, 2\ell+2, 2ikr) = \text{real} \\
&\sim \frac{1}{k^{\ell+1}} |F_\ell(k)| \sin \left( kr - \eta \log 2kr - \frac{\pi}{2} \ell + \sigma_\ell \right) \quad (r \rightarrow \infty) . \quad (\text{A}\cdot 1)
\end{aligned}$$

Here,  $\eta$  is the Sommerfeld parameter and  $F_\ell(k)$  is the Coulomb Jost function

$$F_\ell(k) = e^{\frac{\pi}{2}\eta} \frac{\ell!}{\Gamma(\ell+1+i\eta)} , \quad (\text{A}\cdot 2)$$

which can be obtained by comparing the behavior at the origin between  $\psi_\ell(k, r)$  and  $\varphi_\ell(k, r)$ . The Jost solution of the Coulomb problem is defined by the irregular solution with the asymptotic behavior  $f_\ell(k, r) \sim e^{i(kr - \eta \log 2kr - \pi\ell/2)}$  for  $r \rightarrow \infty$ . More explicitly, it is given by

$$\begin{aligned}
f_\ell(k, r) &= (-i)^\ell (2kr)^{-i\eta} e^{ikr} G(\ell+1+i\eta, -\ell+i\eta, 2ikr) \\
&= i(-)^{\ell+1} e^{\pi\eta/2} (2kr)^{\ell+1} e^{ikr} \Psi(\ell+1+i\eta, 2\ell+2, -2ikr) \\
&\sim e^{i(kr - \eta \log 2kr - \pi\ell/2)} \quad (r \rightarrow \infty) . \quad (\text{A}\cdot 3)
\end{aligned}$$

Here,  $G(\alpha, \beta, z)$  and  $\Psi(\alpha, \gamma, z)$  are irregular solutions of the confluent hypergeometric functions defined in Refs. 22) and 23), respectively, and they are related to each other by

$$\Psi(\alpha, \gamma, z) = z^{-\alpha} G(\alpha, \alpha - \gamma + 1, -z) . \quad (\text{A}\cdot 4)$$

The symmetries of the Jost solution and the Jost function are given by

$$f_\ell^*(k, r) = (-)^\ell e^{\pi\eta} f_\ell(-k, r) , \quad F_\ell^*(k) = e^{\pi\eta} F_\ell(-k) , \quad (\text{A}\cdot 5)$$

with the Coulomb factor  $e^{\pi\eta}$ . They satisfy the usual definition of the Jost function

$$F_\ell(k) = \lim_{r \rightarrow 0} \frac{(kr)^\ell}{(2\ell-1)!!} f_\ell(k, r) , \quad (\text{A}\cdot 6)$$

and the relationship

$$\varphi_\ell(k, r) = \frac{1}{2ik^{\ell+1}} \{F_\ell^*(k) f_\ell(k, r) - F_\ell(k) f_\ell^*(k, r)\} , \quad (\text{A}\cdot 7)$$

for real  $k$ .

The usual Coulomb wave functions are defined as the real functions satisfying the asymptotic behavior

$$\begin{aligned}
F_\ell(k, r) &\sim \sin \left( kr - \eta \log 2kr - \frac{\pi}{2} \ell + \sigma_\ell \right) , \\
G_\ell(k, r) &\sim \cos \left( kr - \eta \log 2kr - \frac{\pi}{2} \ell + \sigma_\ell \right) , \quad (\text{A}\cdot 8)
\end{aligned}$$



for  $r \rightarrow \infty$ . These Coulomb wave functions are related to each other through

$$\begin{aligned}\psi_\ell(k, r) &= \frac{1}{k} e^{i\sigma_\ell} F_\ell(k, r) , \\ \varphi_\ell(k, r) &= \frac{1}{k^{\ell+1}} |F_\ell(k)| F_\ell(k, r) = \text{real} , \\ f_\ell(k, r) &= e^{-i\sigma_\ell} [G_\ell(k, r) + iF_\ell(k, r)] , \\ f_\ell^*(k, r) &= e^{i\sigma_\ell} [G_\ell(k, r) - iF_\ell(k, r)]\end{aligned}\quad (\text{A}\cdot 9)$$

The relationship with the usual “incident plane wave + outgoing (or incoming) spherical wave” is  $\psi_\ell^{(+)}(k, r) = \psi_\ell(k, r)$  and  $\psi_\ell^{(-)}(k, r) = \psi_\ell^*(k, r)$ . This implies that

$$\begin{aligned}\psi_\ell^{(+)}(k, r) &= \psi_\ell(k, r) \sim \frac{1}{k} e^{i\sigma_\ell} \sin \left( kr - \eta \log 2kr - \frac{\pi}{2} \ell + \sigma_\ell \right) , \\ &\sim \frac{1}{k} \sin \left( kr - \eta \log 2kr - \frac{\pi}{2} \ell \right) + f_\ell^C e^{i(kr - \eta \log 2kr - (\pi/2)\ell)} , \\ \psi_\ell^{(-)}(k, r) &= \psi_\ell^*(k, r) \sim \frac{1}{k} e^{-i\sigma_\ell} \sin \left( kr - \eta \log 2kr - \frac{\pi}{2} \ell + \sigma_\ell \right) , \\ &\sim \frac{1}{k} \sin \left( kr - \eta \log 2kr - \frac{\pi}{2} \ell \right) + f_\ell^{C*} e^{-i(kr - \eta \log 2kr - (\pi/2)\ell)} .\end{aligned}\quad (\text{A}\cdot 10)$$

Here,  $f_\ell^C = (1/2ik)(e^{2i\sigma_\ell} - 1)$  is the Coulomb partial-wave amplitude, and

$$e^{2i\sigma_\ell} = \frac{\Gamma(\ell + 1 + i\eta)}{\Gamma(\ell + 1 - i\eta)} , \quad |F_\ell(k)| = \left[ \frac{e^{2\pi\eta} - 1}{2\pi\eta} \prod_{n=1}^{\ell} \frac{n^2}{n^2 + \eta^2} \right]^{\frac{1}{2}} . \quad (\text{A}\cdot 11)$$

## Appendix B

### —— Shift function of various screening functions ——

In this appendix, we calculate the shift function

$$\zeta^\rho(k) \equiv \frac{1}{2k} \int_{\frac{1}{2k}}^{\infty} \frac{2k\eta}{r} \alpha_\rho(r) dr = \eta \int_{\frac{1}{2k}}^{\infty} \frac{1}{r} \alpha_\rho(r) dr , \quad (\text{B}\cdot 1)$$

appearing in Eq. (1.5) for various screening functions  $\alpha^\rho(r)$  and evaluate the no-screening limit  $\rho \rightarrow \infty$ . When the screening is  $\alpha_\rho(r) = e^{-(r/\rho)^n}$ , we can write an analytic expression

$$\zeta^\rho(k) = \eta \int_{\frac{1}{2k}}^{\infty} \frac{1}{r} e^{-\left(\frac{r}{\rho}\right)^n} dr = \eta \log(2k\rho) - \frac{\eta}{n} \gamma + \frac{\eta}{n} \sum_{r=1}^{\infty} \frac{(-)^r}{r r!} \left( \frac{1}{2k\rho} \right)^{nr} , \quad (\text{B}\cdot 2)$$

which leads to

$$\zeta^\rho(k) \rightarrow \eta \log(2k\rho) - \frac{\eta}{n} \gamma \quad \text{as } \rho \rightarrow \infty . \quad (\text{B}\cdot 3)$$

Here,  $\gamma$  is the Euler constant. On the other hand, the screening functions with more sharp transitions like 3)' in Eq. (4.5) seem to have no constant term like Eq. (6.13)

in the limit of  $\rho \rightarrow \infty$ . We will show this for the error function screening in Eq. (6·10). The proof for the exponential screening function in Eq. (C·7) is also carried out similarly.

In order to prove Eq. (6·13), we separate the  $r$  integral in Eq. (B·1) into three pieces as

$$\begin{aligned}\zeta^\rho(k) &= \eta \int_{\frac{1}{2k}}^\rho \frac{1}{r} dr - \eta \int_{\frac{1}{2k}}^\rho \frac{1}{r} (1 - \alpha_\rho(r)) dr + \eta \int_\rho^\infty \frac{1}{r} \alpha_\rho(r) dr \\ &= \eta \log(2k\rho) - I_1(\rho) + I_2(\rho) .\end{aligned}\quad (\text{B}\cdot 4)$$

First, the positive integral  $I_2(\rho)$  is estimated by

$$I_2(\rho) < \frac{\eta}{\rho} \int_\rho^\infty \alpha_\rho(r) dr , \quad (\text{B}\cdot 5)$$

so that we only need to evaluate the integral over  $\alpha_\rho(r)$ . For the error function screening, the expression

$$\alpha_\rho(r) = \frac{1}{\sqrt{\pi}} \left\{ \int_{\beta(r-\rho)}^\infty e^{-t^2} dt + \int_{\beta(r+\rho)}^\infty e^{-t^2} dt \right\} \quad (\text{B}\cdot 6)$$

yields

$$\int_\rho^\infty \alpha_\rho(r) dr = \frac{1}{\sqrt{\pi}} \frac{1}{2\beta} \left( 1 + e^{-(2\beta\rho)^2} \right) - \frac{\rho}{2} (1 - \text{erf}(2\beta\rho)) . \quad (\text{B}\cdot 7)$$

We therefore find

$$I_2(\rho) < \frac{1}{\sqrt{\pi}} \frac{\eta}{2\beta\rho} \left( 1 + e^{-(2\beta\rho)^2} \right) - \frac{\eta}{2} (1 - \text{erf}(2\beta\rho)) \longrightarrow 0 \quad \text{as } \rho \rightarrow \infty . \quad (\text{B}\cdot 8)$$

In order to evaluate  $I_1(\rho)$ , we use

$$\alpha_\rho(r) = 1 - \frac{1}{\sqrt{\pi}} \int_{\beta(\rho-r)}^{\beta(\rho+r)} e^{-t^2} dt \quad (\text{B}\cdot 9)$$

derived from Eq. (B·6), and express it as

$$I_1(\rho) = \frac{\eta}{\sqrt{\pi}} \int_{\frac{1}{2k}}^\rho \frac{1}{r} \left( \int_{\beta(\rho-r)}^{\beta(\rho+r)} e^{-t^2} dt \right) dr . \quad (\text{B}\cdot 10)$$

Here, we change the integral variable from  $r$  to  $x$  by  $r = \rho x$  and obtain

$$I_1(\rho) = \frac{\eta}{\sqrt{\pi}} \int_\varepsilon^1 \frac{1}{x} \left( \int_{\alpha(1-x)}^{\alpha(1+x)} e^{-t^2} dt \right) dx , \quad (\text{B}\cdot 11)$$

with  $\alpha = \beta\rho$  and  $\varepsilon = \frac{1}{2k\rho}$ . We consider the upper bound  $\frac{\eta}{\sqrt{\pi}} \tilde{I}_1(\alpha) > I_1(\rho)$  with

$$\tilde{I}_1(\alpha) = \int_0^1 \frac{1}{x} \left( \int_{\alpha(1-x)}^{\alpha(1+x)} e^{-t^2} dt \right) dx . \quad (\text{B}\cdot 12)$$

We separate the integral interval  $[0, 1]$  into  $[0, 1-\delta]$  and  $[1-\delta, 1]$  with a small positive  $\delta > 0$ . Then, we find

$$\begin{aligned} \tilde{I}_1(\alpha) = & \int_0^{1-\delta} \frac{1}{x} \left( \int_{\alpha(1-x)}^{\alpha(1+x)} e^{-t^2} dt \right) dx \\ & + \int_{1-\delta}^1 \frac{1}{x} \left( \int_{\alpha(1-x)}^{\alpha(1+x)} e^{-t^2} dt \right) dx . \end{aligned} \quad (\text{B}\cdot 13)$$

Here, the first term is bounded by  $2\alpha e^{-(\alpha\delta)^2}(1-\delta)$ . As to the second term, we change the integral variable from  $x$  to  $y$  by  $x = 1 - y$  and find

$$\begin{aligned} \text{2-nd term} = & \int_0^\delta \frac{1}{1-y} \left( \int_{\alpha y}^{\alpha(2-y)} e^{-t^2} dt \right) dy \\ < & \frac{1}{1-\delta} \int_0^\delta \left( \int_0^\infty e^{-t^2} dt \right) dy = \frac{\delta}{1-\delta} \frac{\sqrt{\pi}}{2} . \end{aligned} \quad (\text{B}\cdot 14)$$

Thus, we obtain

$$0 \leq \tilde{I}_1(\alpha) \leq 2\alpha e^{-(\alpha\delta)^2}(1-\delta) + \frac{\delta}{1-\delta} \frac{\sqrt{\pi}}{2} . \quad (\text{B}\cdot 15)$$

First, we take the limit  $\alpha \rightarrow \infty$  in Eq. (B·15) and obtain

$$0 \leq \lim_{\alpha \rightarrow \infty} \tilde{I}_1(\alpha) \leq \frac{\delta}{1-\delta} \frac{\sqrt{\pi}}{2} . \quad (\text{B}\cdot 16)$$

Since we can take  $\delta > 0$  arbitrary small, we eventually find

$$I_1(\rho) \leq \frac{\eta}{\sqrt{\pi}} \tilde{I}_1(\alpha) \longrightarrow 0 \quad \text{as } \rho \rightarrow \infty . \quad (\text{B}\cdot 17)$$

### Appendix C

—— Screening function  $\alpha^\rho(R)$  for the  $dp$  scattering ——

In this appendix, we derive the screening function  $\alpha^\rho(R)$  for the  $pd$  scattering, starting from the screened Coulomb function  $\omega^\rho(r; 1, 2)$  in Eq. (6·15) for the  $pp$  system of the quark-model baryon-baryon interaction. We first note that the  $\rho \rightarrow \infty$  limit,  $\omega = \lim_{\rho \rightarrow \infty} \omega^\rho$ , is an error function Coulomb, which satisfies

$$\langle \psi_d | (P\omega) | \psi_d \rangle \sim \frac{e^2}{R} \quad \text{for } R \rightarrow \infty , \quad (\text{C}\cdot 1)$$

where  $\langle 1, 2 | \psi_d \rangle$  is the deuteron wave function and  $P$  is the rearrangement permutation operator  $P = P_{(12)}P_{(23)} + P_{(13)}P_{(23)}$ . We follow the procedure similar to the  $\alpha\alpha$  case in Eq. (6·9) and separate the folding  $pd$  potential in Eq. (6·16) for the screened Coulomb force into the long-range and short-range parts:

$$V_{pd}^{\rho C}(R) = \langle \psi_d | (P\omega^\rho) | \psi_d \rangle = \langle \psi_d | (P\omega^\rho) - (P\omega) | \psi_d \rangle + \langle \psi_d | (P\omega) | \psi_d \rangle$$

$$\begin{aligned}
&= \left( \frac{e^2}{R} - \langle \psi_d | (P\omega) - (P\omega^\rho) | \psi_d \rangle \right) + \left( -\frac{e^2}{R} + \langle \psi_d | (P\omega) | \psi_d \rangle \right) \\
&= \frac{e^2}{R} \alpha^\rho(R) + \mathcal{W}(R) = W^\rho(R) + \mathcal{W}(R) .
\end{aligned} \tag{C.2}$$

Here, the screening function  $\alpha^\rho(R)$  and the short-range Coulomb potential  $\mathcal{W}(R)$  is given by

$$\begin{aligned}
\alpha^\rho(R) &= 1 - \frac{R}{e^2} \langle \psi_d | (P\omega) - (P\omega^\rho) | \psi_d \rangle , \\
\mathcal{W}(R) &= \langle \psi_d | (P\omega) - \frac{e^2}{R} | \psi_d \rangle .
\end{aligned} \tag{C.3}$$

On the other hand, the exchange term Eq. (5.2) in the three-body model space yields the matrix element

$$V_{pd}^{\rho C}(R) = \langle \psi_d | \mathcal{W}^\rho + W^\rho | \psi_d \rangle = \langle \psi_d | \mathcal{W}^\rho | \psi_d \rangle + W^\rho(R) . \tag{C.4}$$

We therefore find that the deuteron matrix element of the polarization potential is  $\rho$ -independent:

$$\langle \psi_d | \mathcal{W}^\rho | \psi_d \rangle = \mathcal{W}(R) . \tag{C.5}$$

We first assume the sharply cut-off Coulomb force Eq. (5.1) with  $\alpha_\rho(r) = \theta(\rho - r)$  for the two protons, and examine the screening property discussed in §4 by using available analytic expressions. This is possible, if we further neglect the  $D$ -state component of the deuteron wave function and assume that the spatial part of the  $S$ -wave component is given by a simple exponential function  $u(r) = \sqrt{2\gamma} e^{-\gamma r}$ . In this case, the folding potential is expressed in terms of the integral exponential function defined by

$$\begin{aligned}
\text{Ei}(-x) &= - \int_x^\infty \frac{e^{-t}}{t} dt \\
&= \log x + \gamma - x + \frac{x^2}{2 \cdot 2!} - \cdots + \frac{(-x)^r}{r \cdot r!} - \cdots < 0 \quad (\text{for } x > 0) \\
&\sim e^{-x} \sum_{n=1}^\infty (-)^n \frac{(n-1)!}{x^n} \quad (\text{asymptotic expansion}) .
\end{aligned} \tag{C.6}$$

We find

$$\begin{aligned}
V_{pd}^{\rho C}(R) &= \mathcal{W}(R) + \frac{e^2}{R} \alpha_\rho(R) , \\
\mathcal{W}(R) &= V_{pd}^C(R) - \frac{e^2}{R} = -\frac{e^2}{R} e^{-4\gamma R} - 4\gamma e^2 \text{Ei}(-4\gamma R) \\
&\sim -e^{-4\gamma R} \frac{e^2}{4\gamma R^2} \left( 1 - \frac{2!}{4\gamma R} + \frac{3!}{(4\gamma R)^2} - \cdots \right) ,
\end{aligned}$$

$$\begin{aligned}\alpha_\rho(R) &= 1 - 2\gamma \int_\rho^\infty dr [\text{Ei}(-4\gamma(r+R)) - \text{Ei}(-4\gamma|r-R|)] \\ &= -2\gamma \left[ \int_{R-\rho}^\infty dr \text{Ei}(-4\gamma r) + \int_{R+\rho}^\infty dr \text{Ei}(-4\gamma r) \right] \quad \text{for } R \geq \rho. \quad (\text{C}\cdot 7)\end{aligned}$$

Here,  $V_{pd}^C(R) = \lim_{\rho \rightarrow \infty} V_{pd}^{\rho C}(R) = \langle \psi_d | (P\omega) | \psi_d \rangle$ . In order to derive the last expression of Eq. (C.7), we use the relationship

$$(-4\gamma) \int_0^\infty dr \text{Ei}(-4\gamma r) = 1, \quad (\text{C}\cdot 8)$$

which is obtained by exchanging the integration order. From here, we can obtain

$$(-2\gamma) \int_\rho^\infty dr \text{Ei}(-4\gamma|r-R|) = 1 + 2\gamma \int_{R-\rho}^\infty dr \text{Ei}(-4\gamma r) \quad \text{for } R \geq \rho. \quad (\text{C}\cdot 9)$$

The asymptotic form of  $\mathcal{W}(R)$  in Eq. (C.7) is due to

$$(-4\gamma) \text{Ei}(-4\gamma r) \sim \frac{1}{r} e^{-4\gamma r} \left( 1 - \frac{1}{4\gamma r} + \frac{2!}{(4\gamma r)^2} - \dots \right) \quad \text{as } r \rightarrow \infty. \quad (\text{C}\cdot 10)$$

If we further use this in the last expression of Eq. (C.7), we find

$$\begin{aligned}\alpha_\rho(R) &\sim -\frac{1}{2} [\text{Ei}(-4\gamma(R-\rho)) + \text{Ei}(-4\gamma(R+\rho))] \\ &\sim -\text{Ei}(-4\gamma R) \rightarrow 0 \quad \text{as } R \rightarrow \infty.\end{aligned} \quad (\text{C}\cdot 11)$$

With  $R$  fixed, we can show  $\lim_{\rho \rightarrow \infty} \alpha_\rho(R) = 1$  as follows. First, Eq. (C.9) and some calculations yield

$$\begin{aligned}&\frac{2\gamma}{R} \int_0^\infty dr [\text{Ei}(-4\gamma(r+R)) - \text{Ei}(-4\gamma|r-R|)] \\ &= \frac{1}{R} + \frac{4\gamma}{R} \int_R^\infty dr \text{Ei}(-4\gamma r) = \frac{1}{R} (1 - e^{-4\gamma R}) - 4\gamma \text{Ei}(-4\gamma R).\end{aligned} \quad (\text{C}\cdot 12)$$

Thus, Eq. (C.10) gives

$$\begin{aligned}&2\gamma \int_0^\infty dr [\text{Ei}(-4\gamma(r+R)) - \text{Ei}(-4\gamma|r-R|)] \\ &\sim 1 - e^{-4\gamma R} \frac{1}{4\gamma R} \left( 1 - \frac{2!}{4\gamma R} + \dots \right) \quad \text{as } R \rightarrow \infty.\end{aligned} \quad (\text{C}\cdot 13)$$

Here, because of  $r+R \geq |r-R|$ , the integrand of Eq. (C.13) is always positive. Furthermore, the integral from 0 to  $\rho$  in Eq. (C.13) is the monotonically increasing function of  $\rho$  and the limit  $\rho \rightarrow \infty$  exists. We therefore find

$$\lim_{\rho \rightarrow \infty} 2\gamma \int_\rho^\infty dr [\text{Ei}(-4\gamma(r+R)) - \text{Ei}(-4\gamma|r-R|)] = 0. \quad (\text{C}\cdot 14)$$

After all, we find that the screening function  $\alpha^\rho(R)$  satisfies the condition 1) - 3) at least in this simplest case. If we calculate the shift function  $\zeta^\rho(k)$  using  $\alpha^\rho(R)$  in Eq. (C·7), we find the same result Eq. (6·13); namely, there is no constant term as in the sharply cut-off Coulomb case.

The calculation of  $\alpha^\rho(R)$  using the screened Coulomb potential in Eq. (6·15) and the realistic deuteron wave function by the quark-model baryon-baryon interaction is rather involved. We here show only the final result for the numerical calculations. The screening function  $\alpha^\rho(R)$  in the channel-spin formalism is given by

$$\alpha_{(\ell S_c),(\ell' S'_c)}^J(R) = \delta_{\ell,\ell'} \delta_{S_c,S'_c} - \sum_{\lambda,\lambda'=0,2} \sum_{\kappa=0,2,4} f_{\lambda\lambda'}^\kappa(R) g_{(\ell S_c),(\ell' S'_c)}^{\lambda\lambda'\kappa J}, \quad (\text{C} \cdot 15)$$

where the kinematical factor  $g_{(\ell S_c),(\ell' S'_c)}^{\lambda\lambda'\kappa J}$  is given by

$$\begin{aligned} g_{(\ell S_c),(\ell' S'_c)}^{\lambda\lambda'\kappa J} &= (-)^{S_c+S'_c+1} 3 \widehat{S}_c \widehat{S}'_c \widehat{\lambda} \widehat{\ell} \widehat{\lambda}' \widehat{\ell}' \langle \lambda 0 \lambda' 0 | \kappa 0 \rangle \langle \ell 0 \ell' 0 | \kappa 0 \rangle \\ &\times \sum_S (2S+1) \left\{ \begin{array}{ccc} \frac{1}{2} & 1 & S_c \\ \lambda & S & 1 \end{array} \right\} \left\{ \begin{array}{ccc} \frac{1}{2} & 1 & S'_c \\ \lambda' & S & 1 \end{array} \right\} \\ &\times \sum_L (-)^L (2L+1) \left\{ \begin{array}{ccc} \lambda & \ell & L \\ \ell' & \lambda' & \kappa \end{array} \right\} \left\{ \begin{array}{ccc} J & \ell & S_c \\ \lambda & S & L \end{array} \right\} \left\{ \begin{array}{ccc} J & \ell' & S'_c \\ \lambda' & S & L \end{array} \right\}. \quad (\text{C} \cdot 16) \end{aligned}$$

The spatial function  $f_{\lambda\lambda'}^\kappa(R)$  is given by

$$\begin{aligned} f_{\lambda\lambda'}^\kappa(R) &= \int_0^\infty dr u_\lambda(r) u_{\lambda'}(r) v_\kappa(R, r/2), \\ v_\kappa(R, r/2) &= \frac{1}{2} \int_{-1}^1 dx v \left( \sqrt{R^2 + r^2/4 - rRx} \right) P_\kappa(x), \\ v(r) &= \frac{R}{2r} \left[ \text{erf} \left( \frac{\sqrt{3} r + \rho}{2} \frac{r}{b} \right) + \text{erf} \left( \frac{\sqrt{3} r - \rho}{2} \frac{r}{b} \right) \right], \quad (\text{C} \cdot 17) \end{aligned}$$

where  $u_\lambda(r)$  is the  $S$ -wave ( $\lambda = 0$ ) and  $D$ -wave ( $\lambda = 2$ ) deuteron wave functions usually denoted by  $u(r)$  and  $w(r)$ , respectively.

### References

- 1) W. Glöckle, J. Golak, R. Skibiński and H. Witała Phys. Rev. C **79** (2009), 044003.
- 2) W. Glöckle, J. Golak, R. Skibiński and H. Witała Few-Body Systems **47** (2010), 3.
- 3) R. Skibiński, J. Golak, H. Witała and W. Glöckle Eur. Phys. J. A **40** (2009), 215.
- 4) H. Witała, R. Skibiński, J. Golak and W. Glöckle Eur. Phys. J. A **41** (2009), 369.
- 5) H. Witała, R. Skibiński, J. Golak and W. Glöckle Eur. Phys. J. A **41** (2009), 385.
- 6) C. M. Vincent and S. C. Phatak, Phys. Rev. C **10** (1974), 391.
- 7) A. Deltuva, A. C. Fonseca and P. U. Sauer, Phys. Rev. C **71** (2005), 054005.
- 8) A. Deltuva, A. C. Fonseca and P. U. Sauer, Phys. Rev. C **72** (2005), 054004.
- 9) A. Deltuva, A. C. Fonseca, A. Kievsky, S. Rosati, P. U. Sauer and M. Viviani, Phys. Rev. C **71** (2005), 064003.
- 10) E. O. Alt, W. Sandhas, H. Zankel and H. Ziegelmann, Phys. Rev. Lett. **37** (1976), 1537.
- 11) E. O. Alt and W. Sandhas, Phys. Rev. C **21** (1980), 1733.
- 12) E. O. Alt, W. Sandhas and H. Ziegelmann, Nucl. Phys. A **445** (1985), 429.

- 13) E. O. Alt and W. Sandhas, “Collision Theory for Two- and Three-Particle Systems Interacting via Short-Range and Coulomb Forces” in *Coulomb Interactions in Nuclear and Atomic Few-Body Collisions*, eds. F. S. Levin and D. A. Micha (Plenum Press, New York and London, 1996).
- 14) E. O. Alt, A. M. Mukhamedzhanov, M. M. Nishonov and A. I. Sattarov, Phys. Rev. C **65** (2002), 064613.
- 15) J. R. Taylor, Nuovo Cimento **23B** (1974), 313.
- 16) M. D. Semon and J. R. Taylor, Nuovo Cimento **26A** (1975), 48.
- 17) E. O. Alt, P. Grassberger and W. Sandhas, Nucl. Phys. B **2** (1967), 167.
- 18) M. L. Goldberger and K. M. Watson, *Collision Theory*, (Dover Publications, Inc., Mineola, New York, 1964).
- 19) R. G. Newton, *Scattering Theory of Waves and Particles*, (Dover Publications, Inc., Mineola, New York, 1966).
- 20) J. R. Taylor, *Scattering Theory — The Quantum Theory of Nonrelativistic Collisions —*, (Dover Publications, Inc., Mineola, New York, 1972).
- 21) H. van Haeringen, *Charged-Particle Interaction — Theory and Formulas —*, (Coulomb Press Leyden, 1985).
- 22) L. D. Landau and E. M. Lifshitz, *Quantum Mechanics, Non-relativistic theory — 3rd edition —* (Pergamon Press, Oxford, 1977).
- 23) N. N. Lebedev, *Special Functions and Their Applications*, (Dover Publications, Inc., New York, 1972).
- 24) F. Calogero, *The Variable Phase Approach to Potential Scattering*, (New York, N.Y., 1967).
- 25) Gy. Bencze, Nucl. Phys. A **196** (1972), 135.
- 26) H. Witała, W. Glöckle and H. Kamada, Phys. Rev. C **43** (1991), 1619.
- 27) Y. Fujiwara, T. Fujita, M. Kohno, C. Nakamoto and Y. Suzuki, Phys. Rev. C **65** (2002), 014002.
- 28) Y. Fujiwara and K. Fukukawa, Prog. Theor. Phys. **124** (2010), 433.
- 29) K. Fukukawa and Y. Fujiwara, Prog. Theor. Phys. **125** (2011), 729.
- 30) S. Ishikawa, Phys. Rev. C **80** (2009), 054002, and private communications.
- 31) P. Doleschall, W. Grüebler, V. König, P. A. Schmelzbach, F. Sperisen and B. Jenny, Nucl. Phys. A **380** (1982), 72.
- 32) K. Sagara, H. Oguri, S. Shimizu, K. Maeda, H. Nakamura, T. Nakashima and S. Morinobu, Phys. Rev. C **50** (1994), 576.
- 33) H. Shimizu, K. Imai, N. Tamura, K. Nisimura, K. Hatanaka, T. Saito, Y. Koike and Y. Taniguchi, Nucl. Phys. A **382** (1982), 242.
- 34) H. Witała, W. Glöckle, L. E. Antonuk, J. Arvieux, D. Bachelier, B. Bonin, A. Boudard, J. M. Cameron, H. W. Fielding, M. Garçon, F. Jourdan, C. Lapointe, W. J. McDonald, J. Pasos, G. Roy, I. The, J. Tinslay, W. Tornow, J. Yonnet and W. Ziegler, Few-Body Systems **15** (1993), 67.
- 35) Y. Fujiwara and K. Fukukawa, Prog. Theor. Phys. **125** (2011), 979.